For Pre-University and Higher Secondary Students

A First Course in Plane Coordinate Geometry

[With Solid Geometry]

By

K. K. Basu

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A FIRST COURSE IN PLANE COORDINATE GEOMETRY

[With Solid Geometry]

● For Pre-University, University Entrance and Higher Secondary Students. ●

By

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PREFACE TO THE THIRD EDITION

This edition is practically a reprint of the previous edition ; only an alternative method of derivation of the equations of tangents to Conic Sections avoiding the use of differential notation has been introduced.

I take this opportunity of expressing my heartiest thanks to my numerous friends and students serving in different colleges and schools of the province for the warm reception they have accorded to this book.

July, 1966.

K. K. Basu,

PREFACE TO THE SECOND EDITION

The need for a new edition has afforded me an opportunity to thoroughly revise the book and to make a few additions and alterations in conformity with the syllabus prescribed for the Pre-University course. A chapter on Transformation of Coordinates involving transfer of origin without rotation of axes has accordingly been introduced and the principle applied in reducing the equations of the Conic Sections to their standard forms. The discussion of the properties of Poles and Polars has been left out as it is considered beyond the scope of the syllabus. The differential notation has however been used in deriving the equations of tangents to curves as it much simplifies the work and at the same time helps the student understand better the mode of formation of the equations directly from the definition. Two new chapters containing discussion of the properties of Lines and Planes in space as required by the portion of the syllabus relating to Solid Geometry have also been incorporated in the present edition.

I gratefully acknowledge my indebtedness to my numerous friends and colleagues, particularly to Prof. D. Mallik, M.Sc., of Dum Dum Matijheel College and Prof. N. Ghosh, M.Sc., of Burdwan Raj College for the kind interest they have taken and for their valuable suggestions and comments.

Berhampore,
September, 1960.

K. K. Basu.

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SOLID GEOMETRY

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Equation of a locus in rectangular Cartesian co-ordinates. Transfer of origin without rotation of axes. Equations of a straight line in different forms. Angle between two straight lines, conditions for parallelism and perpendicularity. Perpendicular distance of a point from a given line. Equations of the angle-bisectors between two lines. Equation of a circle. Intersection of a straight line and a circle. Condition of tangency. Equations of the tangent and the normal at a point.

Definition of a conic with reference to focus and directrix : a parabola, an ellipse and a hyperbola. Deduction, from definition, of the equations of the above loci referred to the directrix and the perpendicular from the focus upon the directrix as axes. Reduction of these equations to their standard forms. Intersection of a straight line with any of the above loci ; condition for tangency. Equation of tangent and normal at a point for each of the above loci. Deduction of simple properties of the above loci.

(*b*) *Solid Geometry* :

Definitions—Parallel and skew straight lines. Angle between two straight lines and between two planes, parallelism and perpendicularity. Angle between a plane and a straight line, their parallelism and perpendicularity. Projection of a line on a plane. Axioms—(*i*) One and only one plane passes through a given line and a given point outside it. (*ii*) If two planes have one point in common they have at least a second point in common. Theorems—(*i*) Two intersecting planes cut one another in a straight line and in no point outside it. (*ii*) If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane in which they lie. (*iii*) All straight lines drawn perpendicular to a given straight line at a given point on it are coplanar. (*iv*) If, of two parallel straight lines, one is perpendicular to a plane, the other is also perpendicular to it. (*v*) If a straight line is perpendicular to a plane, then every plane passing through it is also perpendicular to that plane.

Idea of the following regular solids : sphere, rectangular parallelopiped, regular tetrahedron, right prism, circular cylinder and a right cone. Expressions (without proof) for the surfaces and volumes of the above solids.

SYLLABUS FOR HIGHER SECONDARY COURSE

Co-ordinate Geometry :

Rectangular Cartesian co-ordinates in a plane. Lengths of segments. Sections of a finite segment in a given ratio. Area of a triangle. Straight line.

Circle : Chords, Tangents, Normals and elementary properties connected with them ; Parabola, Ellipse, Hyperbola referred to their principal axes ; Analytical treatment of these curves in respect of (1) the focus and the directrix properties, (2) tangents and normals and elementary properties connected with them, (3) centre and diameter (Note : Discussion should always be restricted to rectangular Cartesian co-ordinates.)

Solid Geometry :

Axiom (*i*) : One and only one plane may be made to pass through any two intersecting straight lines.

Axiom (*ii*) : Two intersecting planes cut one another in a straight line and in no point outside it.

To prove

1. If a straight line is perpendiculr to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

2. All straight lines drawn perpendicular to a given straight line at a given point of it are coplanar.

3. If two straight lines are parallel, and if one of them is perpendicular to a plane, the other is also perpendicular to the plane.

Concept of angle between two planes and angle between a straight line and a plane. Concept of parallelism of planes. Concept of a line being parallel to a plane. Concept of skew lines.

Mensuration :

Parallelopipeds, Right circular Cones, Prisms and Pyramids (Expressions, without proof, of the surfaces and volumes of these solids).

PART ONE

PLANE COORDINATE GEOMETRY

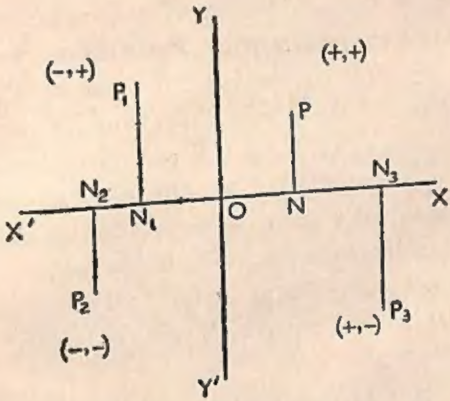
CHAPTER I

COORDINATES, LENGTHS OF SEGMENTS, AREAS

I-1. Coordinates :

To fix the position of a point in a plane we must refer it to some fixed points or lines in the plane. It has been found that the simplest plan is to take two straight lines intersecting at right angles in the plane as lines of reference to which the position of the point on the plane may be referred.

Let XOX' and YOY' be two straight lines intersecting at right angles at O and let P be the given point.



Draw PN parallel to OY to meet OX in N. The position of the point P is definitely known when we know the distances ON and NP and the directions in which they are drawn.

With regard to the directions the usual convention is followed —that is to say, ON is taken positive when drawn to the right from O and negative when drawn to the left from O ; while NP is taken positive if drawn upwards from N and negative if drawn downwards.

The lengths ON and NP with their proper signs are called **Coordinates** of the point. ON is called the **abscissa** or the x-coordinate and NP the **Ordinate** or the y-coordinate of the point P. If ON is 'x' units of length and NP 'y' units and drawn as in the figure, *i.e.*, if the abscissa and ordinate of P be x and y respectively, then P is known as the point (x, y). The fixed lines are called **axes of Coordinates** ; XOX', the axis of x and YOY' the axis of y and O is called the **Origin**.

The two axes divide the plane into four parts called quadrants. For any point in the first quadrant XOY both the abscissa and the ordinate are positive, in the second quadrant YOX' the abscissa is negative and the ordinate is positive, in the third quadrant $X'OY'$ both the abscissa and the ordinate are negative while in the fourth quadrant $Y'OX$ the abscissa is positive and the ordinate negative.

Given the position of a point in the plane of the axes, we know how to get the coordinates of the point. Conversely, if the coordinates (x, y) of a point P are given, we can determine its position by first measuring ON equal to 'x' units of length along the x-axis and then from N drawing NP equal to 'y' units parallel to the y axis, both ON and NP being drawn in their proper directions as indicated by the sign of the coordinates. The point P we finally arrive at is the position of the point (x, y). For example, to locate the position of the point $(-2, 3)$ we take N_1 on OX' such that ON_1 is equal to 2 units and then draw N_1P_1 parallel to OY equal to 3 units of length. The point P_1 is the point $(-2, 3)$.

It will be at once seen that a point lying on the x-axis will have its ordinate equal to zero, and that a point lying on the y-axis will have its abscissa equal to zero, while the Coordinates of the origin are $(0, 0)$.

The system of coordinates referred to above is known as **Cartesian (rectangular)** Coordinates after the name of **Des Cartes** who first introduced this system. There are also other systems of coordinates but we shall be concerned only with this system in this elementary volume.

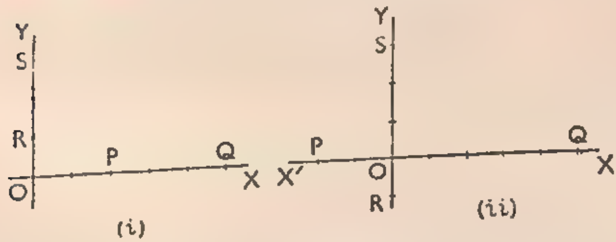
I-2. Oblique Axes :

When the axes of coordinates are inclined to each other at an angle other than a right angle the axes are said to be *oblique*. In oblique axes also the convention regarding the signs as well as the mode of fixing the position of a point is the same as in the case of rectangular axes. When oblique, the angle between the positive directions of the axes is generally denoted by ω.

As it is generally more convenient to take the axes at right angles, we shall, throughout this book, assume the axes to be rectangular unless otherwise stated.

I-3. Distance between two points situated on an axis :

If two points P and Q are taken on the x-axis where



$OP=x_1$ and $OQ=x_2$, the length PQ is given by x_2-x_1, whether the points are on the same or opposite sides of the origin, provided in substituting numerical values we attach proper signs.

In fig. (*i*) where $OP=x_1=2$ and $OQ=x_2=5$

$$x_2-x_1=5-2=3=PQ\,;$$

In fig. (*ii*) where $OP=x_1=-2$ and $OQ=x_2=5$

$$x_2-x_1=5-(-2)=5+2=7=PQ$$

so that in both cases the length $PQ=x_2-x_1$.

Similarly, if two points R and S are taken on the y-axis where $OR=y_1$ and $OS=y_2$ the length RS is given by the expression y_2-y_1 whatever may be the positions of the points on the axis of y.

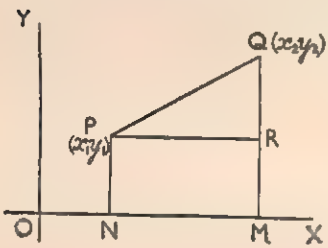
The above result holds for rectangular as well as oblique axes.

Remark : To avoid confusion regarding signs we shall always consider points or lines in the first quadrant and the conclusions thus arrived at will be equally true for their positions in any other quadrant and hence the results may be treated as perfectly general.

I-4. Length of the line joining two given points :

(a) *Rectangular Axes.*

Let the two given points be $P\ (x_1, y_1)$ and $Q\ (x_2, y_2)$.



Draw the ordinates PN and QM and draw PR parallel to OX to meet QM in R.

Then

$$PR = NM = OM - ON = x_2 - x_1$$

$$RQ = MQ - MR = MQ - NP = y_2 - y_1.$$

$$\therefore \quad PQ^2 = PR^2 + RQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

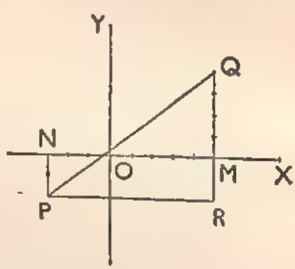
Hence, $\mathbf{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$.

The formula has been proved for the case when both the points are in the first quadrant. It will be found to be true whatever may be the positions of the points.

As an example, if the two points are $P\ (-3, -2)$ and $Q\ (5, 4)$ so that $x_1 = -3$, $y_1 = -2$ and $x_2 = 5$, $y_2 = 4$ then $PR = NO + OM = 3 + 5 = -x_1 + x_2$ and $RQ = RM + MQ = 2 + 4 = -y_1 + y_2$.

Hence, $PQ = \sqrt{PR^2 + RQ^2}$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



Cor. The distance of a point (α, β) from the origin $(0, 0)$ is given by $\sqrt{\alpha^2 + \beta^2}$.

(b) *Oblique Axes.*

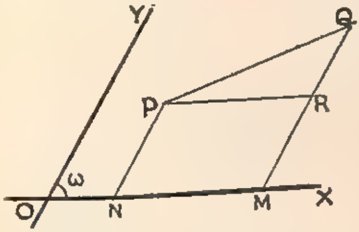
In this case the ordinates NP and MQ are also inclined to the x-axis at an angle ω

$$\therefore \quad \angle PRQ = \pi - \omega.$$

We have $PQ^2 = PR^2 + RQ^2 - 2PR.RQ \cos PRQ$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

$+ 2(x_2 - x_1)(y_2 - y_1) \cos \omega,$



since, as before,

$PR = x_2 - x_1$, $RQ = y_2 - y_1$

and $\cos(\pi - \omega) = -\cos \omega$.

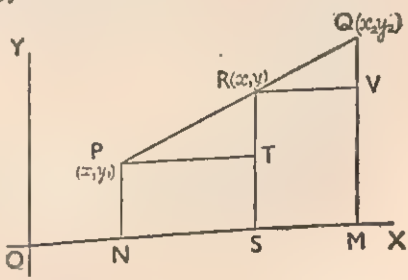
$$\therefore\ PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega}$$

Note : When the axes are rectangular, $\omega = 90°$.

$\therefore$ $\cos\ \omega = 0$ and the expression for the distance reduces to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, as has already been derived.

I-5. Section of a line in a given ratio :

To find the coordinates of the point which divides in a given ratio $(m : n)$ the line joining the points $(x_1 y_1)$ and $(x_2 y_2)$.



Case I. Internal division: Let $R\ (x, y)$ be the point which divides the line joining $P\ (x_1 y_1)$ and $Q (x_2 y_2)$ in the given ratio, so that

$$PR : RQ = m : n.$$

Draw the ordinates PN, QM and RS and draw PT, RV parallel to OX meeting RS and QM in T and V respectively.

Then from similar triangles PTR and RVQ

we have (*i*) $PT : RV = PR : RQ$

and (*ii*) $TR : VQ = PR : RQ$

From (*i*) since $PT = NS = OS - ON = x - x_1$

and $RV = SM = OM - OS = x_2 - x$

we have $\dfrac{x - x_1}{x_2 - x} = \dfrac{m}{n}$,

so that $n(x - x_1) = m(x_2 - x)$

i.e., $x(m+n)=mx_2+nx_1$

and hence, $x=\dfrac{mx_2+nx_1}{m+n}$.

Again from (*ii*) since $TR=SR-ST=SR-NP=y-y_1$

and $VQ=MQ-MV=MQ-SR=y_2-y$

we have $\dfrac{y-y_1}{y_2-y}=\dfrac{m}{n}$

so that $n(y-y_1)=m(y_2-y)$

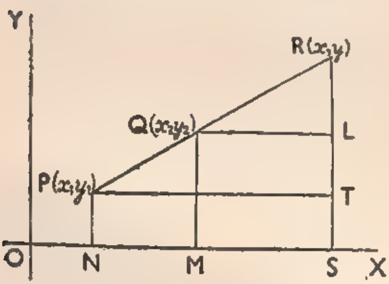
and hence, $y=\dfrac{my_2+ny_1}{m+n}$.

Hence, the coordinates of the point which divides the line joining (x_1y_1) and (x_2y_2) **internally** in the ratio $m:n$ are

$$\frac{\mathbf{mx_2+nx_1}}{\mathbf{m+n}} \text{ and } \frac{\mathbf{my_2+ny_1}}{\mathbf{m+n}} \qquad \cdots \qquad (A)$$

Case II. External division :

If R divides PQ externally in the ratio $m:n$ so that $PR:QR=m:n$, we have from the figure, considering the two similar triangles PTR and QLR,



(*i*) $\dfrac{PT}{QL}=\dfrac{PR}{QR}$, and

(*ii*) $\dfrac{TR}{LR}=\dfrac{PR}{QR}$.

Hence, from (*i*) $\dfrac{x-x_1}{x-x_2}=\dfrac{m}{n}$

giving $x=\dfrac{mx_2-nx_1}{m-n}$

and from (*ii*) $\dfrac{y-y_1}{y-y_2}=\dfrac{m}{n}$

giving $y=\dfrac{my_2-ny_1}{m-n}$.

Hence, the coordinates of the point which divides the line joining (x_1y_1) and (x_2y_2) **externally** in the ratio $m:n$ are

$$\frac{\mathbf{mx_2-nx_1}}{\mathbf{m-n}} \text{ and } \frac{\mathbf{my_2-ny_1}}{\mathbf{m-n}} \qquad \cdots \qquad (B)$$

Note : The expressions (*B*) can be obtained by writing $-n$ for n in (*A*). This can be explained in the following way :

Since $QR=-RQ$ algebraically, therefore $\frac{PR}{RQ}=\frac{m}{-n}$; hence, to get the coordinates of R in *Case II* we have only to change the sign of n in the expressions already found in *Case I.*

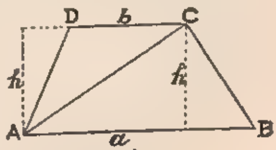
Cor. If $m=n$ in *Case I*, R is the middle point of PQ and hence the coordinates of the mid-point of the line joining (x_1y_1) and (x_2y_2) are $\frac{x_1+x_2}{2}$ and $\frac{y_1+y_2}{2}$, *i.e.*, the Arithmetic mean between the abscissa and the ordinates of the given points.

The results derived in this article are clearly true for *rectangular as well as for oblique axes.*

I-6. Area of a triangle :

Lemma : *The area of a trapezium is equal to half the sum of the parallel sides multiplied by the perpendicular distance between them.*

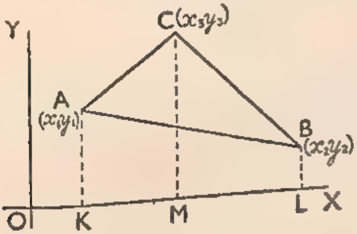
If a and b be the lengths of the parallel sides AB and DC respectively of the trapezium $ABCD$ and h be the distance between these sides, then from the adjoining figure, joining the diagonal AC we have



$$\text{Area } ABCD=\triangle ABC+\triangle ACD$$
$$=\tfrac{1}{2}ah+\tfrac{1}{2}bh$$
$$=\tfrac{1}{2}(a+b)\,h.$$

To find the area of a triangle, having given the coordinates of its angular points.

(*a*) *Rectangular axes.*



Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the coordinates of the vertices A, B, C of the triangle ABC. Draw the ordinates AK, BL, CM. Then

$\triangle ABC=$trapezium $AKMC$ $+$trape. $CMLB-$trape. $AKLB$

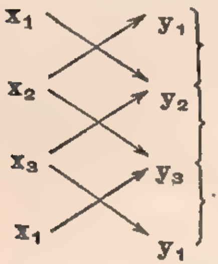
$$=\tfrac{1}{2}\,KM\,(KA+MC)+\tfrac{1}{2}\,ML\,(MC+LB)-\tfrac{1}{2}\,KL\,(KA+LB)$$
$$=\tfrac{1}{2}(x_3-x_1)(y_1+y_3)+\tfrac{1}{2}(x_2-x_3)(y_3+y_2)-\tfrac{1}{2}(x_2-x_1)(y_1+y_2).$$

Hence, on simplification,

$$\triangle ABC = \tfrac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3]$$

Cor. The area of the triangle OBC (one vertex is at the origin is $\frac{1}{2}(x_2y_3 - x_3y_2)$.

Remark 1. The following working rule for writing down the expression for the area of a triangle in terms of the coordinates of the vertices may be usefully employed. Write down the coordinates of the vertices in a column and repeat the first set at the end. Multiply across as indicated by the arrows and attach positive signs to the products obtained by descending and negative signs to those obtained by ascending. Finally, add up and take half the sum.



$$x_1y_2 - x_2y_1$$

$$x_2y_3 - x_3y_2$$

$$x_3y_1 - x_1y_3$$

Remark 2. The expression for the area of a triangle is positive if the vertices are taken in the counter-clockwise order, *i.e.*, in the order in which a person supposed to travel round the triangle finds the area always to his left. If the vertices be taken in the clockwise order, the area will be found to be negative, so that in this case we must change the sign of the expression of Art. I-6, in order to get the area as a positive quantity as it necessarily is.

Remark 3. The expression for the area may be written as

$$\tfrac{1}{2}(x_2y_3 - x_3y_2) + \tfrac{1}{2}(x_3y_1 - x_1y_3) - \tfrac{1}{2}(x_2y_1 - x_1y_2)$$

so that we may interpret the result as expressing

$$\begin{aligned}\triangle ABC &= \triangle OBC + \triangle OCA - \triangle OBA \\ &= \triangle OBC + \triangle OCA + \triangle OAB. \qquad [\textit{Cor., Art. I-6}]\end{aligned}$$

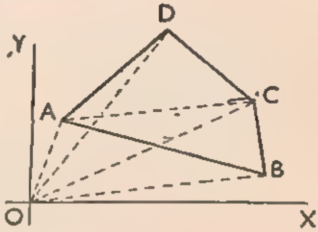
(*b*) *Oblique axes.*

With the same construction as before, it is found that the ordinates AK, BL, CM not being perpendiculars to OX, the distances between the parallel sides are not KM, ML and KL,

but are equal respectively to $KM \sin\omega$, $ML \sin\omega$ and $KL \sin\omega$, so that the expression for the area in this case becomes

$$\tfrac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3] \sin\omega.$$

I-7. Area of quadrilateral or polygon :



The area of a quadrilateral $ABCD$ whose vertices taken in the counter-clockwise order are $A(x_1y_1)$ $B(x_2y_2)$, $C(x_3y_3)$, $D(x_4y_4)$ may be deduced from the area of a triangle either by joining a diagonal AC and taking the sum of the areas of the triangles ABC and CDA or by joining O to each of the angular points A, B, C and D and taking quad. $ABCD = \Delta OBC + \Delta OCD + \Delta ODA + \Delta OAB$. [$\because \Delta OAB = -\Delta OBA$]

The area will be found to be

$$= \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)\}$$

In the same way we obtain the area of the polygon whose vertices taken in the counter-clockwise order are (x_1y_1), (x_2y_2)............ (x_ny_n). If $ABCD...K$ be the polygon, then regard being had to the signs of the areas, we have

$$\text{Poly. } ABC...K = \Delta OAB + \Delta OBC + \Delta OCD + \ldots\ldots + \Delta OKA$$
$$= \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots\ldots + (x_ny_1 - x_1y_n)\}.$$

Observe that the working rule for writing down the expression for the area of the triangle also applies to the case of quadrilateral and polygon.

Note : The expressions are to be multiplied by $\sin\omega$ to get the areas in oblique coordinates.

WORKED OUT EXAMPLES

Ex. 1. *Find the centre and radius of the circle circumscribing the triangle whose vertices are* (8,4), (7,7) *and* (3,9).

Let the centre be (α, β) and the radius, r. Then since the length of the line joining the centre to each of the given points is equal to the radius, we have

$$(\alpha-8)^2 + (\beta-4)^2 = r^2 \quad \cdots \quad \cdots \quad (1)$$
$$(\alpha-7)^2 + (\beta-7)^2 = r^2 \quad \cdots \quad \cdots \quad (2)$$
$$(\alpha-3)^2 + (\beta-9)^2 = r^2 \quad \cdots \quad \cdots \quad (3)$$

From (1) and (2), $(\alpha-8)^2+(\beta-4)^2=(\alpha-7)^2+(\beta-7)^2$

$$i.e., \quad \alpha-3\beta+9=0 \qquad \cdots \qquad \cdots \qquad (4)$$

Similarly, from (2) and (3)

$$2\alpha-\beta-2=0 \qquad \cdots \qquad \cdots \qquad (5)$$

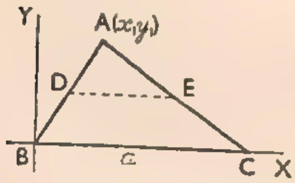
Solving (4) and (5), we get $\alpha=3$, $\beta=4$.

Substituting these values of α and β in any one of the relations (1), (2) or (3) we find $r^2=25$, $\therefore \ r=5$.

Hence, the centre is at the point (3, 4) and the radius equal to 5.

Ex. 2. *Prove that the line joining the mid-points of any two sides of a triangle is half the third side.*

Let ABC be the triangle and D, E the mid-points of AB, AC respectively.



Take B as the origin, BC as the axis of x and a line through B perpendicular to BC as the axis of y.

Let $BC=a$, so that C is the point $(a, 0)$, and let the vertex A be (x_1, y_1).

Then the points D and E are respectively

$$\left(\frac{x_1}{2}, \frac{y_1}{2}\right) \text{ and } \left(\frac{x_1+a}{2}, \frac{y_1}{2}\right).$$

Hence, $$DE^2=\left(\frac{x_1+a}{2}-\frac{x_1}{2}\right)^2+\left(\frac{y_1}{2}-\frac{y_1}{2}\right)^2=\left(\frac{a}{2}\right)^2.$$

Therefore, $DE=\frac{a}{2}$ i.e., $DE=\frac{1}{2}BC$.

Ex. 3. *D, E, F are the middle points of the sides BC, CA, AB respectively of a triangle ABC; prove that the point G which divides AD internally in the ratio 2 : 1 also divides BE and CF in the same ratio.*

Hence prove that the medians of a triangle are concurrent and find the point of concurrence. [Draw a figure]

Let the vertices A, B, C be (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively. Then if D be the point (X_1, Y_1), we have

$$X_1=\tfrac{1}{2}(x_2+x_3), \ Y_1=\tfrac{1}{2}(y_2+y_3).$$

Let G be the point $(\bar{x}, \bar{y})$;

Since, G divides the join of (x_1, y_1) and (X_1, Y_1) in the ratio 2:1, we have

$$\bar{x}=\frac{2X_1+1.x_1}{2+1}=\frac{2.\frac{1}{2}(x_2+x_3)+x_1}{3}=\frac{x_1+x_2+x_3}{3};$$

$$\bar{y}=\frac{2Y_1+1.y_1}{2+1}=\frac{2.\frac{1}{2}(y_2+y_3)+y_1}{3}=\frac{y_1+y_2+y_3}{3}.$$

Proceeding in the same way it can be shown that these are also the coordinates of the points which divide BE and CF in the ratio 2 : 1.

Hence, the point G which lies on the median AD also lies on the other medians BE and CF. It therefore follows that the medians of a triangle are concurrent, the point of concurrence *i.e.*, the **centroid** being

$$\left(\frac{\mathbf{x_1+x_2+x_3}}{\mathbf{3}}, \frac{\mathbf{y_1+y_2+y_3}}{\mathbf{3}}\right)$$

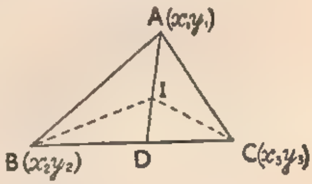
where (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of the triangle.

Ex. 4. *Prove that the coordinates of the in-centre of the triangle whose vertices are* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *are*

$$\frac{ax_1+bx_2+cx_3}{a+b+c} \text{ and } \frac{ay_1+by_2+cy_3}{a+b+c}$$

where a, b, c *are respectively the sides opposite to the vertices.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC. Let the bisectors of the angles A, B and C meet at I. AI is produced to meet BC in D.



We have

$BD : DC = AB : AC = c : b$.

Hence, D is the point $\left(\frac{cx_3+bx_2}{c+b}, \frac{cy_3+by_2}{c+b}\right)$.

Again, $\frac{AI}{ID}=\frac{AC}{CD}$ and also $\frac{AI}{ID}=\frac{AB}{BD}$.

$$\therefore \frac{AI}{ID}=\frac{AC+AB}{CD+BD}=\frac{b+c}{a}.$$

Hence, the coordinates $\bar{x}$ and $\bar{y}$ of I, the in-centre are given by

$$\bar{x}=\frac{(b+c)\times\dfrac{cx_3+bx_2}{c+b}+ax_1}{b+c+a} \text{ and } \bar{y}=\frac{(b+c)\times\dfrac{cy_3+by_2}{c+b}+ay_1}{b+c+a}$$

i.e., $\bar{x}=\dfrac{ax_1+bx_2+cx_3}{a+b+c}$ and $\bar{y}=\dfrac{ay_1+by_2+cy_3}{a+b+c}$.

EXERCISE I

1. Find the distance between the following pairs of points :

(*i*) (1, 2) and (4, 6) ; (*ii*) $(-3, 4)$ and $(3, -4)$;

(*iii*) (5, 7) and $(-7, 2)$; (*iv*) $(a, -b)$ and $(-a, b)$;

(*v*) (0, 0) and $(a \cos \alpha, a \sin \alpha)$;

(*vi*) $(a \cos \theta, a \sin \theta)$ and $(a \cos \phi, a \sin \phi)$.

2. Prove that the triangle whose vertices are $(-2, 5)$, $(5, -2)$ and (10, 10) is isosceles.

3. Prove that the points $(-3, 5)$, $(6, -1)$, (10, 5) are the vertices of a right-angled triangle and find the length of the hypotenuse.

4. Find the centre and radius of the circle circumscribing the triangle whose vertices are (1, 5), $(7, -13)$ and $(17, -3)$

5. Prove that the four points (1, 5), (2, 4), (3, 1) and $(-2, 6)$ lie on a circle and find the centre and radius of this circle.

6. Prove that the points $(2a, 4a)$, $(2a, 6a)$ and $(2a+\sqrt{3}a, 5a)$ are the vertices of an equilateral triangle whose side is $2a$. [*C. U.* 1952]

7. Prove that the points $(-2, 6)$, (1, 2), (10, 4) and (7, 8) when joined in order form a parallelogram.

8. Prove that the four points $(-2, 1)$, $(-1, -3)$, $(3, -2)$ and (2, 2) are the vertices of a square.

9. If the figure formed by joining the four points (α_1, β_1), (α_2, β_2), (α_3, β_3) and (α_4, β_4), taken in order, be a parallelogram, then prove that

$$\alpha_1+\alpha_3=\alpha_2+\alpha_4 \text{ and } \beta_1+\beta_3=\beta_2+\beta_4.$$

10. Find the coordinates of the point which divides the join of (1, 8), (6, 3) in the ratio 2 : 3.

11. Find the coordinates of the point which divides (*i*) internally, (*ii*) externally the line joining the points $(-10, 4)$ and $(-2, -4)$ in the ratio 5 : 3.

12. Find the coordinates of the points of trisection of the line joining $(1, -3)$ and (4, 6).

13. Find the ratio in which the line joining the points $(2, -3)$ and $(7, 5)$ is divided by the axis of x.

14. Find the centroid of the triangle whose vertices are $(1, 4)$, $(5, -3)$ and $(6, 8)$.

15. If the centroid of a triangle is $(1, 4)$ and two of its vertices are $(4, -3)$ and $(-9, 7)$ find the other vertex.

16. In any triangle ABC prove that

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

where D is the middle point of BC.

[**Hints :** Take B as the origin and BC as the x-axis.]

17. (*i*) If G be the centroid of the triangle ABC prove that

$$GA^2 + GB^2 + GC^2 = \tfrac{1}{3}(BC^2 + CA^2 + AB^2).$$

(*ii*) Prove that the lines joining the middle points of opposite sides of a quadrilateral and the line joining the middle points of its diagonals meet in a point and bisect one another [*C. U.*]

18. If the distance of the point P whose coordinates are (x, y) from the point $(3, 4)$ is equal to 5, find the relation between x and y.

19. If the point $P(x, y)$ is equidistant from the two points (a, b), (b, a), find the relation between x and y.

20. Find the area of the triangle whose vertices are

(*i*) $(0, 0)$, $(5, 3)$, $(1, 9)$;

(*ii*) $(-2, 3)$, $(6, 2)$, $(4, 7)$;

(*iii*) $(-3, -4)$, $(6, 3)$, $(-1, 5)$;

(*iv*) $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$.

21. Prove that the following sets of three points are in a straight line :

(*i*) $(5, -3)$, $(9, 5)$ and $(11, 9)$;

(*ii*) $(-4, 4)$, $(2, 0)$ and $(5, -2)$;

(*iii*) $(0, -\frac{1}{2})$, $(\frac{1}{3}, 0)$ and $(\frac{2}{3}, \frac{1}{2})$.

[**Hints :** Show that the area of the triangle formed by joining the points is zero.]

22. Show that the three points $(4, 2)$, $(7, 5)$, $(9, 7)$ lie on a right line. [*C. U.*]

23. Prove that the condition that the three points whose coordinates are (x_1, y_1), (x_2, y_2) and (x_3, y_3) should be collinear is

$$x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0.$$

24. If D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle ABC, then prove that

$$\Delta ABC = 4 . \Delta DEF.$$

25. Find the area of the quadrilateral, whose angular points are $(1, 1), (3, 4), (5, -2), (4, -7)$. [C. U.]

26. Show that the area of the quadrilateral whose vertices, taken in order, are $(a, 0), (-b, 0), (0, a), (0, -b)$, is zero ; $(a, b > 0)$. Explain the result with the help of a diagram. [C. U.]

27. The coordinates of A, B, C are $(6, 3)$, $(-3, 5)$ and $(4, -2)$ respectively and P is the point (x, y) ; show that

$$\frac{\Delta PBC}{\Delta ABC}=\frac{x+y-2}{7}. \qquad [C.\ U.]$$

Answers :

1. (*i*) 5 ; (*ii*) 10 ; (*iii*) 13 ; (*iv*) $2\sqrt{a^2+b^2}$; (*v*) a ; (*vi*) $2a \sin \frac{\theta \sim \phi}{2}$. **3.** 13. **4.** $(7, -3)$; 10. **5.** $(-2, 1)$; 5. **10.** $(3, 6)$. **11.** $(-5, -1)$: $(10, -16)$. **12.** $(2, 0)$; $(3, 3)$. **13.** $(3 : 5)$. **14.** $(4, 3)$, **15.** $(8, 8)$. **18.** $x^2+y^2-6x-8y=0$. **19.** $x=y$. **20.** (*i*) 21 ; (*ii*) 19 ; (*iii*) $23\frac{1}{4}$; (*iv*) $a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$. **25.** $20\frac{1}{2}$.

CHAPTER II

LOCUS AND EQUATION

II-1. Locus : Equation to a locus :

Hitherto we had been considering representation of isolated points. We now proceed to investigate a method by which an assemblage of points all obeying some fixed law can be represented.

Locus : A locus is the path (straight or curved) described by a point which moves so as always to satisfy a given condition or conditions.

For instance, the locus of a point which moves in a plane so as always to be at a constant distance from a given point is a circle. Again, the locus of a point which moves so as always to be at a constant distance from a fixed straight line is another straigth line parallel to the given straight line.

Locus and Equation : Consider the equation

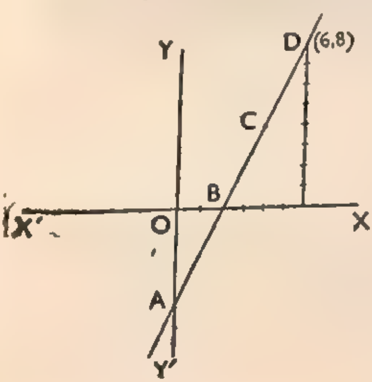
$$2x-y=4 \quad \cdots \quad \cdots \quad (1)$$

From this equation we can get an infinite number of pairs of values of x and y which sa fy the equation. The following are some of the pairs :

$x=0$ $\quad$ $x=2$ $\quad$ $x=4$ $\quad$ $x=-2$ $\quad$ $x=-4$

$y=-4$ $\quad$ $y=0$ $\quad$ $y=4$ $\quad$ $y=-8$ $\quad$ $y=-12$

Each pair of values is taken to represent the coordinates of a point. We thus get a number of points A $(0,-4)$, $B\,(2, 0)$, $C\,(4, 4)$·········etc. These points are now plotted on the squared paper and joined.



We get a straight line which, we say, is the graph or the locus of the equation (1), and (1) is said to be the equation to the straight line AB.

Again, consider the equation $x^2+y^2=25$ ... (2)

If we perform the same operation with this equation, we shall find a circle whose centre is at the origin and radius equal to 5. We say that the circle is the locus of the equation (2), and (2) will be said to be the equation to the circle.

We observe that—

(1) x and y represent the coordinates of **any** point on the graph thus drawn.

(2) If *any* point is taken on the graph its coordinates will be found to satisfy the equation ; for example, a point D on the graph of equation (1) is found to have coordinates 6 and 8 and $x=6$, $y=8$ clearly satisfy the equation.

(3) The coordinates of any point not on the locus do not satisfy the equation.

The **equation to a curve** may therefore be defined as the algebraical relation between the coordinates x and y of **any point** on the curve and which holds for no point outside it.

Conversely, corresponding to every equation in x and y there is, in general, a curve the coordinates of every point of which satisfy the equation. This curve is called the **Locus of the equation.**

Note : In Coordinate Geometry the word 'curve' is used to mean a line (straight or curved) or a set of such lines satisfying given geometrical conditions.

When a point is capable of taking up different positions on a curve it is said to be a **current point** and its coordinates are said to be **current coordinates.** In Cartesian Coordinates, it is usually denoted by (x, y). The coordinates of a point fixed in a particular position are taken as (h, k), (a, β), (x', y'), (x_1, y_1) etc.

In Coordinate Geometry we shall be given the geometrical condition or conditions under which a point moves and thereby describes a certain locus and we shall be required to find the equation to this locus. We shall find that this geometrical condition can invariably be expressed as an algebraical relation

between the current coordinates of any point on the locus. **This relation between the coordinates is the required equation to the locus.** In dealing with locus problems, the following procedure is generally adopted :

Take $P(x, y)$ any point on the locus. Then from the problem ascertain the geometrical condition or conditions which must be satisfied by this point in order that it may be a point on the locus. Now express this condition in terms of x and y. The relation between the coordinates x and y thus obtained, on simplification gives the equation to the locus required.

To avoid chances of confusion, often (h, k) or (α, β) etc. is taken to represent *any* point on the locus and a relation between these coordinates is established and finally these are substituted by the current coordinates x and y to give the desired equation.

II-2. Illustrative Examples :

A few examples to illustrate the principle of formation of equation to a locus are given below :

Example 1. *Find the locus of a point which moves so that with reference to a pair of rectangular axes, twice its abscissa always exceeds the ordinate by* 4.

Let (x, y) be any point on the locus. From the given condition, $2x$ exceeds y by 4, *i.e.*, $2x-y=4$.

This being the relation between the coordinates of any point on the locus is the required equation to the locus.

[The locus of the equation is shown in the figure of Art. II-1. Take a few points at random on the line and verify by actual measurement that the condition holds for these points.]

Example 2. *Find the locus of a point which moves so that its distance from the origin is always equal to its distance from the point* (3, 0).

Let (x, y) be any point on the locus. The distances of (x, y) from the origin $(0, 0)$ and from the point $(3, 0)$ are respectively

$$\sqrt{x^2+y^2} \quad \text{and} \quad \sqrt{(x-3)^2+y^2}.$$

Hence, from the given condition,

$$\sqrt{x^2+y^2}=\sqrt{(x-3)^2+y^2}.$$

Squaring and simplifying, we get

$$2x-3=0$$

which is the required equation to the locus.

Example 3. *A point moves so that its distance from the point* (1, 0) *is always equal to its distance from the axis of y. Find the equation to its locus.*

Let (x, y) be any point which satisfies the given condition. We then have

$$\sqrt{(x-1)^2+y^2}=x.$$

Squaring and simplifying, we get

$$y^2-2x+1=0,$$

i.e., $$y^2=2x-1.$$

This being the relation between the coordinates of any point that satisfies the given condition, is the required equation to the locus.

Example 4. *Find the locus of a point which moves so that the sum of the squares of its distances from the points* (2, 0) *and* (−2, 0) *is always equal to* 40.

Let the given points be $A(2, 0)$ and $B(-2, 0)$ and let $P(x, y)$ be any point on the locus.

From the given condition, we have

$$PA^2+PB^2=40$$

i.e., $$\{(x-2)^2+y^2\}+\{(x+2)^2+y^2\}=40$$

i.e., $$2x^2+2y^2+8=40$$

i.e., $$x^2+y^2=16$$

which is the required equation to the locus.

[Since the distance of the point (x, y) from the origin is given by $\sqrt{x^2+y^2}$ the equation expresses the fact that the point is always at a distance 4 from the origin, in other words, the locus is a circle of radius 4 having its centre at the origin.

EXERCISE II

Find the equation to the locus of a point which moves so that

1. its distance from the x-axis is always equal to its distance from the y-axis ;

2. its distance from the x-axis is always equal to m times its distance from the y-axis ;

3. its distance from the y-axis always falls short of its distance from the x-axis by a constant quantity a ;

4. twice its distance from the x-axis always exceeds three times its distance from the y-axis by 4 ;

5. the sum of its distances from two fixed straight lines at right angles in the plane in which the point moves is always equal to a constant quantity k.

[**Hints :** Take the fixed lines as axes of reference.]

6. A point moves in such a way that its distance from the point (2, 3) is always equal to 4. Find the locus of the point.

7. A point moves so that its distance from the point $(-2, 0)$ always equals twice its distance from the point (2, 0). Find the equation to its locus.

8. Find the locus of a point which moves so that its distance from the point $(a, 0)$ always exceeds its distance from the y-axis by a.

9. A moving point is always equidistant from two fixed points (2, 0) and (6, 4). Find the locus of the moving point.

10. Find the locus of a point which moves so that its distance from the axis of y is double its distance from the point (2, 2). [*C. U.*]

11. Find the equation to the perpendicular bisector of the join of the two points (1, 3) and (5, 5).

12. The points A and B are $(-4, 0)$ and $(-1, 0)$ respectively. A point P moves so that $PA : PB = 2 : 1$. Find the locus of P.

13. The coordinates oi two fixed points A and B are respectively $(-2, 4)$ and (6, 8). A point P moves so that the area oi the triangle PAB is always 10. Find tha equation to the locus of P.

Answers :

1. $x-y=0$. 2. $y=mx$. 3. $y=x+a$. 4. $2y-3x=4$
5. $x+y=k$. 6. $x^2+y^2-4x-6y-3=0$. 7. $3x^2+3y^2-20x+12=0$.
8. $y^2=4ax$. 9. $x+y=6$. 10. $3x^2+4y^2-16x-16y+32=0$.
11. $2x+y=10$. 12. $x^2+y^2=4$. 13. $2y-x=15$.

CHAPTER III

CHANGE OF AXES

III-1. Transformation of Coordinates.

The coordinates of a point being its distances from the axes measured in proper directions will depend upon the position of the axes. If therefore, the point remaining fixed in position, the axes of reference be changed, there will be a corresponding change in the coordinates of the point. It also follows that the equation to a locus, which is an algebraical relation between the coordinates of a current point on the locus, will undergo a change with the change in the position of the axes of reference.

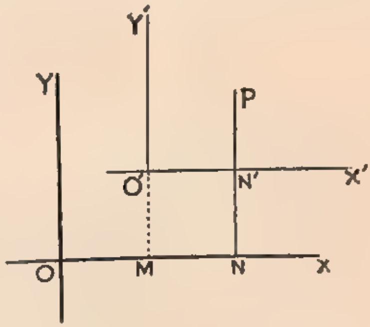
In many problems it will be found desirable to pass from one set of axes to another—a process known as **transformation of Coordinates.** We have therefore to investigate formulæ for such transformation when the axes are changed in any manner—by transferring the origin to a different point keeping the directions of the axes the same as before, or by changing the directions of the axes keeping the origin fixed, or by changing the position of the origin as well as the directions of the axes. We shall, however, at this stage confine ourselves to the most simple case of transformation, *viz.* change of origin without a change in the directions of the axes known as **translation** or **parallel displacement**.

III-2. Transfer of origin keeping the directions of the axes unaltered.

To find the formulæ for transformation from one set of axes to another through a different origin, the new axes being parallel to the original axes, both systems being rectangular.

Let OX, OY be a system of rectangular axes with reference to which the coordinates of any point P are

(x, y) and let $O'X'$, $O'Y'$ be the new axes parallel to OX, OY, the new origin being O' whose coordinates are (h, k). Let the coordinates of the same point P referred to the new axes be (x', y'). Draw PN parrallel to OY meeting $O'X'$ in N' and produce $Y'O'$ to mee OX in M. We then have



$OM=h$, $MO'=k$; $ON=x$,

$NP=y$ and $O'N'=x'$,

$N'P=y'$.

$$\therefore \quad x=ON=OM+MN=OM+O'N'=h+x'$$

$$y=NP=NN'+N'P=MO'+N'P=k+y'$$

Hence, $\mathbf{x=x'+h,\ y=y'+k}$

which give the required formulæ for transformation.

If therefcre, in the equation to a locus, we substitute $x'+h$ for x and $y'+k$ for y we get the transformed equation to the same locus referred to the new axes, x' and y' denoting the current coordinates of any point on the locus with reference to the new axes. If however, the usual convention, *viz.* to denote the current point by (x, y) is followed, the accents in x' and y' are suppressed and the transformed equation is one in x and y. The **Working Rule** to get the transformed equation in this case may thus be laid down :

Write x+h for x and y+k for y in the equation to a locus, where (h, k) are the coordinates of the new origin.

Remark. The above proof clearly applies to **Oblique Axes** as well and consequently we get the same formulæ for transformation in this case also.

III-3. Illustrative Examples.

Example 1. *Transform to parallel axes through the point* $(2, -3)$ *the equation*

$$2x^2+3xy+4y^2+x+18y+25=0.$$

The transformed equation, obtained by writing $x+2$ for x and $y-3$ for y is

$$2(x+2)^2+3(x+2)(y-3)+4(y-3)^2+(x+2)+18(y-3)+25=0$$

which, on reduction, becomes

$$2x^2+3xy+4y^2=1.$$

Example 2. *Transform the equation*

$$6x^2+4xy+3y^2+8x-2y+4=0$$

by referring it to parallel axes through a properly chosen point, so that the transformed equation may be of the form $Ax^2+2Hxy+By^2=1$.

Also find the actual values of A, B *and* H.

Let (h, k) be the new origin, so that the transformed equation is

$$6(x+h)^2+4(x+h)(y+k)+3(y+k)^2+8(x+h)$$
$$-2(y+k)+4=0$$

i.e., $$6x^2+4xy+3y^2+4(3h+k+2)x+2(2h+3k-1)y$$
$$+6h^2+4hk+3k^2+8h-2k+4=0.$$

In order that this may be of the given form in which there is no term containing x and y, we must have

$$3h+k+2=0 \text{ and } 2h+3k-1=0$$

whence $h=-1$ and $k=1$.

$\therefore$ The new origin is $(-1, 1)$.

With these values of h and k, the transformed equation becomes

$$6x^2+4xy+3y^2=1$$

which is of the given form. Comparing it with $Ax^2+2Hxy+By^2=1$, we find

$$A=6,\ B=3,\ H=2.$$

EXERCISE III

1. Transform the following equations by referring to parallel axes through the points indicated against each :

(i) $ax+by+c=0$; $\left(-\frac{c}{a},\ 0\right)$

(ii) $x \cos \alpha + y \sin \alpha = p$; $(p \cos \alpha, p \sin \alpha)$

(iii) $x^2 + y^2 + 2gx + 2fy + c = 0$; $(-g, -f)$.

(iv) $4x^2 + 9y^2 - 40x - 54y + 145 = 0$; $(5, 3)$.

2. Find where the origin is to be transferred without changing the directions of tne axes in order that the equation

$$y^2 + 2y - 8x + 25 = 0$$

may be transformed into the form $y^2 = 4ax$.

Also find the value of a.

3. Find where the origin is to be shifted retaining the directions of the axes so that the terms in x and y may be removed from the equation

$$x^2 - y^2 - 2x + 8y - 15 = 0.$$

Find also the transformed equation.

4. By transferring to parallel axes through a properly chosen point (h, k), prove that the following equation can be reduced to one containing only terms of the second degree ;

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \qquad [C. U.]$$

Answers :

1. (i) $ax + by = 0$; (ii) $x \cos \alpha + y \sin \alpha = 0$;
   (iii) $x^2 + y^2 = g^2 + f^2 - c$; (iv) $4x^2 + 9y^2 = 36$.
2. $(3, -1)$; $a = 2$.
3. $(1, 4)$; $x^2 - y^2 = 0$.
4. $h = -\frac{3}{2}$, $k = -\frac{5}{2}$; the transformed equation is

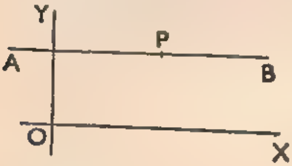
$$12x^2 - 10xy + 2y^2 = 0.$$

CHAPTER IV

THE STRAIGHT LINE

IV-1. Equations of lines parallel to the axes :

To find the equation to a straight line parallel to the axis of x at a distance k from it.



Let AB be the line and $P(x, y)$ any point on it.

Then from the given condition, whatever be the position of the point P on the line, its ordinate is always equal to k, so that

$$\mathbf{y=k}$$

is the required equation to the line.

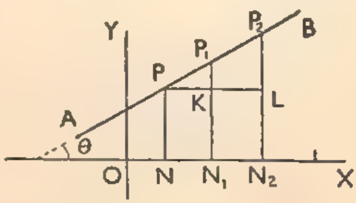
Similarly, the equation to the straight line parallel to the axis of y at a distance h from it is given by

$$\mathbf{x=h}$$

Cor. The equation to the x-axis is $y=0$.
The equation to the y-axis is $x=0$.

IV-2. Gradient of a straight line :

The gradient of a straight line means the increase in the ordinate of a point per unit increase in the abscissa, as the point passes from one position to another on the straight line. P, P_1 are two points on the straight line AB. Draw the ordinates PN, P_1N_1 and draw PK perpendicular to P_1N_1.



If a point is supposed to travel from P to P_1, its ordinate increases by the amount $N_1P_1 - NP$, *i.e.*, KP_1 and the corresponding increase in the abscissa is NN_1, *i.e.*, PK, so that the increase in the ordinate per unit increase in the abscissa is

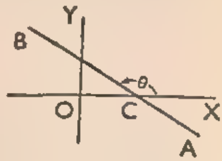
given by the ratio $\frac{KP_1}{PK}$. Hence, the gradient of the line is given by $\frac{KP_1}{PK}$. If the point passes from P to P_2, then from the figure, the gradient is $\frac{LP_2}{PL}$ which is clearly equal to $\frac{KP_1}{PK}$, since the triangles PKP_1 and PLP_2 are similar. It is also clear from the properties of similar triangles that this gradient is the same for all points on the line.

If the same length be taken as unit along both axes, then clearly

$$\textbf{Gradient} = \frac{\mathbf{KP_1}}{\mathbf{PK}} = \tan\theta$$

where θ is the angle between AB and OX.

Hence, *the gradient of a straight line is the same for all points on the line and is equal to the tangent of the angle which it makes with the positive direction of the axis of x.*

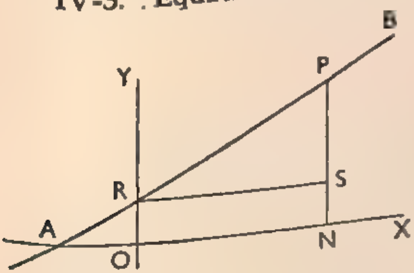


The angle θ which the direction of a straight line makes with the positive direction of OX is the angle through which a line originally parallel to the x axis must turn in the positive (counter-clockwise) sense in order to coincide with the given direction. Thus in the figure, the angle θ which AB makes with OX is $\angle XCB$ and not $\angle OCB$. The angle θ is called the **Slope** of the line.

Cor. Two parallel straight lines having the same slope must have the same gradient.

IV-3. Equation of a straight line : Different forms :

A. Gradient form :



(i) *To find the equation to a straight line having a given gradient and making a given intercept on the axis of y.*

Let AB be the straight line having the given gradient m and cutting the axis of y at R where OR is the given intercept c.

Let $P(x, y)$ be any point on the line. Draw the ordinate PN and draw RS perpendicular to PN. We have then

$$m=\frac{SP}{RS}=\frac{NP-NS}{ON}=\frac{NP-OR}{ON}=\frac{y-c}{x}$$

i.e., $\mathbf{y=mx+c}$

which being the relation between the coordinates of any point on the line is the required equation to the line.

Since $m=\tan\theta$, where θ is the inclination of the straight line to the x-axis, the equation can also be written as

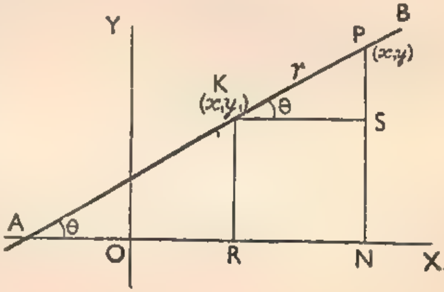
$\mathbf{y=x\tan\theta+c}$.

Cor. The equation to a straight line passing through the origin and having a gradient m (or making an angle θ with the x-axis) is, since the intercept on the axis of y in this case is zero, given by

$\mathbf{y=mx}$ or, $\mathbf{y=x\tan\theta}$.

(*ii*) *To find the equation to a straight line having a given gradient and passing through a given point.*

Let AB be the straight line having the given gradient m and passing through the given point $K(x_1, y_1)$.



Let $P\ (x, y)$ be any point on the line. Draw the ordinates KR, PN and draw KS perpendicular to PN.

Then $$m=\frac{SP}{KS}=\frac{NP-RK}{RN}=\frac{y-y_1}{x-x_1}$$

i.e., $$\mathbf{y-y_1=m(x-x_1)}$$

is the required equation to the line.

Example. *Find the equation to the line through the point* (1, 2) *parallel to the line* $y=\frac{2}{3}x+1$.

Clearly, the gradient of the required line is $\frac{2}{3}$. Hence, the equation is $y-2=\frac{2}{3}(x-1)$.

B. Symmetrical form :

If θ be the slope of the line passing through the given point (x_1, y_1), then the equation of the previous article can be written as

$$y-y_1=\tan\theta\,(x-x_1)$$

i.e.,
$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}.$$

Now let r be the distance from $K(x_1, y_1)$ of any point $P(x, y)$ on the line [*see figure of A (ii)*]. Then

$$r\cos\theta=KS=x-x_1$$

and
$$r\sin\theta=SP=y-y_1$$

so that
$$\mathbf{\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r}$$

is the equation to a line passing through the given point (x_1, y_1) and inclined at an angle θ to the x-axis.

Remark. This form of the equation of a straight line has the particular advantage of giving the coordinates x and y of any point on the line situated at a given distance r from the fixed point (x_1, y_1) on the line. We have

$$\mathbf{x=x_1+r\cos\theta,\quad y=y_1+r\sin\theta.}$$

Example. *Find the distance from* (3, 8) *measured along the line* $4x-3y+12=0$ *to the point where this line intersects the line* $4x+5y=60$.

The equation $4x-3y+12=0$ can be written as $y=\frac{4}{3}x+4$.

If θ be the slope of the line, we have $\tan\theta=\frac{4}{3}$, whence

$$\frac{\sin\theta}{4}=\frac{\cos\theta}{3}=\frac{\sqrt{\sin^2\theta+\cos^2\theta}}{\sqrt{4^2+3^2}}=\frac{1}{5}$$

giving $\cos\theta=\frac{3}{5}$ and $\sin\theta=\frac{4}{5}$.

If r be the required distance of the point of intersection P of the lines from (3, 8), then its coordinates are given by $3+\frac{3}{5}r$ and $8+\frac{4}{5}r$. Since this point lies on the line $4x+5y=60$, we must have $4(3+\frac{3}{5}r)+5(8+\frac{4}{5}r)=60$

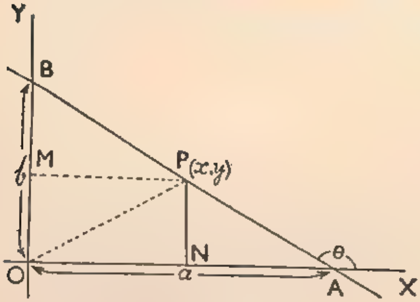
i.e., $(\frac{12}{5}+4)r=60-12-40$, or, $\frac{32}{5}r=8$.

Hence, $r=\frac{5}{4}$ which gives the required distance.

C. Intercept form :

To find the equation to the straight line which cuts off given intercepts from the axes.

Let AB be the straight line which cuts off given intercepts $OA=a$ and $OB=b$ from the axes of x and y respectively and let $P(x, y)$ be any point on the line AB



Draw the ordinate PN. Then from similar triangles,

$$\frac{NA}{OA}=\frac{NP}{OB} \text{ or, } \frac{a-x}{a}=\frac{y}{b}$$

i.e., $\quad bx+ay=ab.$

Hence, dividing both sides by ab, we have

$$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}$$

which is the required equation.

Alternative method :

Join OP and draw PM perpendicular to OB. Then, whatever may be the position of the point P on the line, we must have

$$\triangle BOP+\triangle POA=\triangle BOA$$

i.e., $\quad \frac{1}{2}bx+\frac{1}{2}ay=\frac{1}{2}ab$

whence, $\quad \frac{x}{a}+\frac{y}{b}=1.$

Note 1. The equation can be written as $y=-\frac{b}{a}x+b$. Now comparing this equation with $y=mx+c$, we find that the gradient of the line is given by

$$m=\tan\theta=-\frac{b}{a}.$$

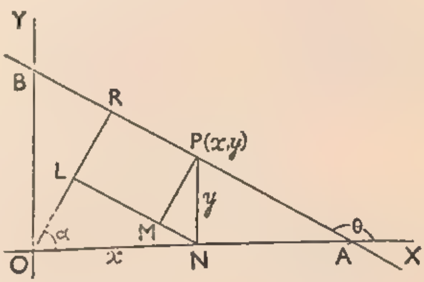
If a and b be both positive, then $\tan\theta$ is negative showing that θ is obtuse, as is clear from the figure.

Note 2. As a, the intercept on the x-axis is given larger and larger values, the line BA becomes more and more nearly parallel to the x-axis, so that when a is made infinitely large, the line becomes parallel to the x-axis and its equation becomes $\frac{y}{b}=1$, *i.e.*, $y=b$ which tallies with the

result of Art. IV-1. Similarly, for a line parallel to the y-axis, the equation is $x=a$.

D. Perpendicular form :

To find the equation to a straight line having given the length of the perpendicular from the origin upon it and the angle which this perpendicular makes with the axis of x.



Let OR be the perpendicular from the origin O, upon the line AB.

Given $OR=p$ and $\angle XOR=\alpha$. To find the equation to the line AB.

Let P (x, y) be any point on the line. Draw the ordinate PN. Also draw NL perpendicular to OR and PM perpendicular to NL. Then

$$OR=OL+LR=OL+MP$$

$$=ON\cos\alpha+NP\sin\alpha$$

(Since, $\angle PNM=90^\circ-\angle ONL=\angle NOL=\alpha$)

i.e., $$\mathbf{x\cos\alpha+y\sin\alpha=p}$$

which being the relation between the coordinates of any point on the line is the required equation to the line.

Note 1. α is measured in the counter-clockwise direction and p is taken positive.

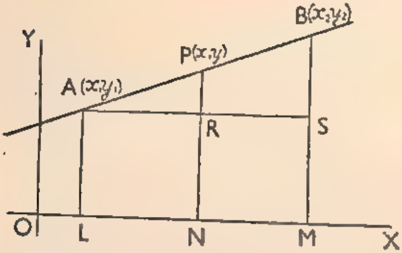
Note 2. (*i*) Writing the equation in the form $y=-\cot\alpha.x+p\operatorname{cosec}\alpha$, we get, the gardient$=-\cot\alpha$, so that $\tan\theta=-\cot\alpha=\tan(90^\circ+\alpha)$ giving $\theta=90^\circ+\alpha$, as is obvious from the figure.

(*ii*) Dividing both sides by p, we have

$$\frac{x\cos\alpha}{p}+\frac{y\sin\alpha}{p}=1 \text{ or, } \frac{x}{p\sec\alpha}+\frac{y}{p\operatorname{cosec}\alpha}=1$$

which is in the intercept form. Hence, the intercepts on the axes are $p\sec\alpha$ and $p\operatorname{cosec}\alpha$.

IV-4. Straight line passing through two given points :



Let $A(x_1, y_1)$, $B(x_2, y_2)$ be the two given points and let $P(x, y)$ be any point on the straight line passing through A and B.

Draw the ordinates AL, BM, PN and draw ARS parallel to OX meeting BM and PN in S and R respectively.

Then from similar triangles ARP, ASB we have

$$\frac{RP}{SB}=\frac{AR}{AS}$$

i.e., $$\frac{NP-LA}{MB-LA}=\frac{ON-OL}{OM-OL}$$

i.e., $$\frac{\mathbf{y-y_1}}{\mathbf{y_2-y_1}}=\frac{\mathbf{x-x_1}}{\mathbf{x_2-x_1}}$$

which being the relation between the coordinates of any point on the line is the required equation to the line.

Otherwise : The gradient of the line is $\frac{y_2-y_1}{x_2-x_1}$. The line is therefore one having a gradient $\frac{y_2-y_1}{x_2-x_1}$ and passing through (x_1,y_1). Hence, by A (*ii*), its equation is $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

i.e., $$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.$$

Cor. The line joining the origin to any given point (x_1, y_1) is $\frac{y}{y_1}=\frac{x}{x_1}$.

IV-5. General equation of a straight line :

From the different forms of the equation to a straight line derived in Art. IV-3 it is found that the equations are all of

the first degree in x and y, *i.e.*, squares and higher powers of x and y do not occur in the equations. We shall now prove that

An equation of the first degree in x and y always represents a straight line.

The most general form of an equation of the first degree in x and y is

$$Ax+By+C=0 \quad \cdots \quad \cdots (1)$$

where A, B and C are constants independent of x and y. This equation represents some geometrical locus and we shall prove that this locus is a straight line.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be any three points on the locus represented by (1).

Then we must have

$$Ax_1+By_1+C=0 \quad \cdots \quad \cdots (2)$$

$$Ax_2+By_2+C=0 \quad \cdots \quad \cdots (3)$$

$$Ax_3+By_3+C=0 \quad \cdots \quad \cdots (4)$$

From (3) and (4),

$$\frac{A}{y_2-y_3}=\frac{B}{x_3-x_2}=\frac{C}{x_2y_3-x_3y_2}=k \text{ (say)} \quad \cdots \quad (5)$$

Substituting for A, B, C from (5) in (2), we have

$$(y_2-y_3)x_1+(x_3-x_2)y_1+(x_2y_3-x_3y_2)=0$$

i.e. $x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3=0$.

Hence, from Art. I-6, $2\Delta PQR=0$ *i.e.*, $\Delta PQR=0$ which shows that P, Q, R must lie on a straight line. But P, Q, R are three points chosen anywhere on the locus of (1). Hence, the result might be taken to express the fact that three points selected anywhere on the locus must lie on a straight line.

This proves that the locus itself is a straight line.

Otherwise: The equation $Ax+By+C=0$ may be written as

$y=-\frac{A}{B}x-\frac{C}{B}$, *i.e.*, $y=mx+c$ where

$m=-\frac{A}{B}$ and $c=-\frac{C}{B}$.

But $y=mx+c$ represents a straight line having gradient m and passing through (o, c). Hence, the equation $Ax+By+C=0$ represents a straight line having gradient $-\frac{A}{B}$ and passing through the point $\left(0, -\frac{C}{B}\right)$. If $B=0$, the equation reduces to $Ax+C=0$ *i.e.* $x=-\frac{C}{A}$ which represents a line parallel to the axis of y.

IV-6. Reduction of the general equation to different forms :

The general equation of a straight line is

$$Ax+By+C=0$$

(1) It can be written as

$$y=-\frac{A}{B}x-\frac{C}{B}$$

which is in the **gradient form.**

(2) To write the equation in the **intercept form** we have

$$Ax+By=-C$$

or, $$\frac{Ax}{-C}+\frac{By}{-C}=1$$

i.e., $$\frac{x}{-C/A}+\frac{y}{-C/B}=1.$$

Hence, the intercepts on the axes are $-\frac{C}{A}$ and $-\frac{C}{B}$.

(3) To reduce the equation to the **perpendicular form**

viz., $x \cos \alpha+y \sin \alpha=p$ ... ... (1).

By multiplying both sides of an equation by a constant quantity we only alter its form and not the equation itself, so that

$$kAx+kBy+kC=0$$

i.e., $-kAx-kBy=kC$ ... ... (2)

is only a new form of the equation $Ax+By+C=0$ (here C is taken to be positive). We have now to find k so that (2) may be the same as (1).

Hence, $\cos \alpha=-kA$, $\sin \alpha=-kB$ ... ... (3)

From (3), squaring and adding

$$k^2A^2+k^2B^2=1$$

giving $k=\frac{1}{\sqrt{A^2+B^2}}$ (taking the positive value .

Hence, the equation (2) becomes

$$-\frac{Ax}{\sqrt{A^2+B^2}}-\frac{By}{\sqrt{A^2+B^2}}=\frac{C}{\sqrt{A^2+B^2}}$$

This is the reduced form of the general equation when C is positive.

If however C is negative, equation (2) is written as

$$kAx+kBy=-kC$$

and the reduced form of the equation will be

$$\frac{Ax}{\sqrt{A^2+B^2}}+\frac{By}{\sqrt{A^2+B^2}}=-\frac{C}{\sqrt{A^2+B^2}}.$$

Remembering that the perpendicular from the origin upon the line is always positive, we have

$$p=\frac{C}{\sqrt{A^2+B^2}} \text{ or, } -\frac{C}{\sqrt{A^2+B^2}}$$

according as C is positive or negative.

The equation $Ax+By+C=0$ is therefore reduced to the perpendicular form by dividing both sides by $\sqrt{A^2+B^2}$ and arranging it so that the numerical term occurs in the right-hand side of the equation as a positive quantity.

Example : If the equation of the straight line be $x+y+\sqrt{2}=0$, we divide by $\sqrt{1^2+1^2}$, *i.e.*, by $\sqrt{2}$ and get $\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}+1=0$, which can be written as $-\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y=1$ *i.e.*, $x\cos 225°+y\sin 225°=1$ which is in the required form.

IV-7. Constants in the equation of a straight line :

From the different forms of the equation to a straight line it is seen that it involves two arbitrary constants. The general equation, *viz.*, $Ax+By+C=0$ though apparently involves three constants A, B and C, in reality contains only two. This can at once be seen if the equation is written as $\frac{A}{C}x+\frac{B}{C}y+1=0$ which is of the form $lx+my+1=0$ involving only two constants l and m.

Now a straight line must satisfy two independent geometrical conditions in order that its position may be fixed, *e.g.*, if a

straight line is to pass through a given point and be parallel to a given straight line then its position is determined. Analytically these two conditions can be expressed as two equations involving the two constants of the straight line. The two constants can now be found out by solving the equations thus obtained and hence the equation to the straight line is uniquely determined.

IV-8. Systems of straight lines :

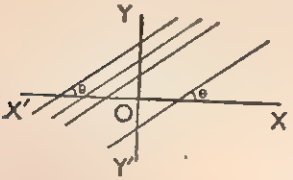
We have seen that the equation to a straight line in any of its forms contains only two arbitrary constants. These constants while remaining the same for the same straight line differ for different straight lines and by suitably choosing them the equation can be made to represent any particular straight line desired.

We now proceed to discuss what the equation represents when one of these constants has a fixed value while the other is capable of taking up different values.

(1) Consider the equation

$$y = mx + c$$

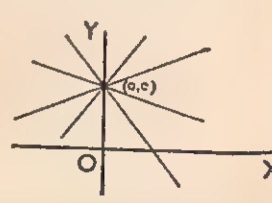
Suppose *m is kept fixed and c is given different values.*



In this case the c's being different we must have different lines but all these lines having the same gradient m must be equally inclined to the x-axis.

Hence, the equation represents *a system of parallel straight lines all inclined at an angle θ to the x-axis where $\tan\theta = m$.*

Suppose now *c is kept fixed but m is given different values.* In this case the m's being different we get lines having different gradients, *i.e.*, drawn in different directions but since c has a fixed value, all these lines must pass through the same point on the axis of y, *viz.*, the point (o, c).

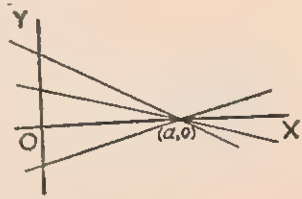


Hence, the equation in this case represents *a system of concurrent straight lines all passing through the point (o, c).*

(2) Consider the equation

$$\frac{x}{a}+\frac{y}{b}=1$$

Suppose *a is kept fixed and b is given different values.*



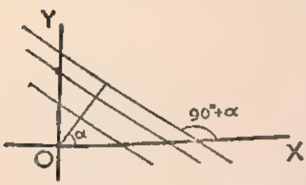
Clearly for different values of b, we have lines cutting off different intercepts from the y-axis but all these lines having the same 'a' cut off the same intercept on the x-axis.

Hence, the equation represents *a system of concurrent straight lines all passing through the point* (a, o).

Similarly, the case of b remaining fixed and a having different values may be treated.

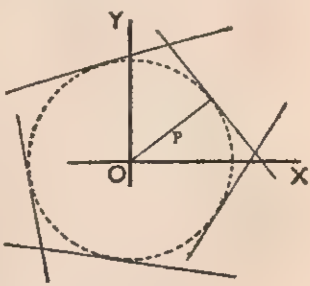
(3) Consider the equation $x \cos \alpha + y \sin \alpha = p$.

Suppose *α is constant and p is given different values.*



Here, p being different, the lengths of the perpendiculars from the origin upon the lines have different values but all these being lines for which α is constant we have the same line perpendicular to all of them so that the lines must be inclined at a constant angle $90^\circ+\alpha$ to the axis of x.

Hence, in this case *the equation represents a system of parallel straight lines all inclined at an angle $90^\circ+\alpha$ to the axis of x.*



Suppose again *p is kept constant but α is given different values.*

Here, the inclinations of the lines to the x-axis will be different for different values of α but p being constant all these lines must be equidistant from the origin and so the feet of the perpendiculars from the origin upon these lines must lie on a circle.

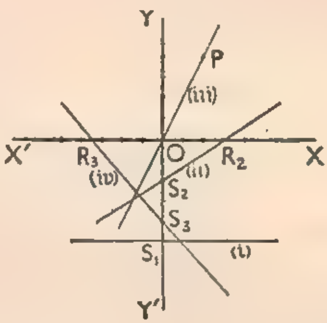
Hence, in this case the equation represents *a system of straight lines all tangents to a fixed circle having its centre at the origin and radius equal to p.*

WORKED OUT EXAMPLES

Ex. 1. *Trace the following straight lines :*

(*i*) $y=-5$; (*ii*) $2x-3y=6$; (*iii*) $y=2x$; (*iv*) $8x+7y+28=0$.

(*i*) By Art. IV-1, $y=-5$ is a straight line parallel to the x-axis and passes through a point S_1 on the axis of y where $OS_1=-5$.



(*ii*) The equation can be written in the form $\frac{x}{3}+\frac{y}{-2}=1$. The intercepts on the axes are thus 3 and -2. If then $OR_2=3$ and $OS_2=-2$, R_2S_2 must be the required line.

(*iii*) The line clearly passes through the origin, since (0, 0) satisfies the equation. Also from the equation, when $x=2$, $y=4$ and so the point P (2, 4) lies on the line. The required line is therefore OP.

(*iv*) Putting $y=0$, we have $x=-\frac{7}{2}$ and putting $x=0$, we get $y=-4$. The points $(-\frac{7}{2}, 0)$ and $(0, -4)$ are therefore on the line. If then $OR_3=-\frac{7}{2}$ and $OS_3=-4$, R_3S_3 must be the required line.

Ex. 2. *Find the equation of the straight line passing through the point* $(-3, 2)$ *and cutting off intercepts equal in magnitude but opposite in sign from the axes.*

Let the intercepts be a and $-a$ so that they are equal but of opposite signs.

The equation of the line is then $\frac{x}{a}+\frac{y}{-a}=1$

i.e., $x-y=a$ ... ... (1)

Since the line passes through the point $(-3, 2)$, these coordinates must satisfy the equation (1).

Hence, $-3-2=a$, or, $a=-5$.

The reqd. equation is thus, on substitution in (1),

$$x-y=-5$$

i.e., $$x-y+5=0.$$

Ex. 3. *A straight line is inclined at an angle* $135°$ *to the axis of* x *and passes through the point* $(3, -4)$. *Find its equation.*

The gradient of the line is given by

$$m=\tan 135°=-1$$

The required equation is therefore

$$y+4=-1\,(x-3) \qquad [\textit{Art. IV-3. A (ii)}]$$

i.e., $$x+y+1=0.$$

Ex. 4. *Find the equation to the straight line which passes through the points* $(5, -2)$ *and* $(-3, 7)$.

The gradient of the line $=\frac{7-(-2)}{-3-5}=-\frac{9}{8}$;

$\therefore$ The required equation is

$$y-(-2)=-\tfrac{9}{8}(x-5)$$

i.e., $$8(y+2)=-9(x-5)$$

or, $$9x+8y=29.$$

Ex. 5. *A straight line passes through the point* $(-4, 9)$ *and is such that the portion of it intercepted between the axes is divided at this point in the ratio* 3:2. *Find its equation.*

Let the required equation be

$$\frac{x}{a}+\frac{y}{b}=1.$$

The points where this line meets the axes are $(a, 0)$ and $(0, b)$.

The point which divides the join of these two points in the ratio 3:2 has coordinates

$$\frac{3.0+2.a}{3+2} \text{ and } \frac{3.b+2.0}{3+2}.$$

i.e., $$\frac{2a}{5} \text{ and } \frac{3b}{5}.$$

But this is the point whose coordinates are -4 and 9.

Hence, $$-4=\frac{2a}{5} \text{ and } 9=\frac{3b}{5}$$

i.e., $$a=-10 \text{ and } b=15.$$

∴ The required equation is

$$\frac{x}{-10}+\frac{y}{15}=1.$$

i.e., $$3x-2y+30=0.$$

Ex. 6. *What does the equation $x \cos \alpha + y \sin \alpha = 5$ represent for different values of α?*

For any given value of α the equation represents a straight line which is such that the length of the perpendicular from the origin upon it is 5 and the perpendicular makes an angle α with the x-axis. Hence, for different values of α, we get different lines but all these lines are at a distance 5 from the origin. The feet of the perpendiculars from the origin on the lines therefore lie on a circle.

Hence, for different values of α, the equation represents a system of lines all tangents to the circle having its centre at the origin and radius equal to 5.

EXERCISE IV(A)

1. State the gradients of the lines passing through the following pairs of points :

(*i*) (1, 2) and (3, 4) ;
(*ii*) (3, -5) and (5, 9) ;
(*iii*) (0, -5) and (-4, 7) ;
(*iv*) (-1, -3) and (4, 6) ;
(*v*) (-5, 4) and (3, 4).

2. Find the angles at which the lines joining the following pairs of points are inclined to the axis of x :

(i) $(-1, 5)$ and $(2, 8)$; (ii) $(0, -5)$ and $(4, -9)$;
(iii) $(-3, -5)$ and $(4, -5)$; (iv) $(0, 5)$ and $(0, -3)$.

3. Find the gradients of the following lines and also the coordinates of the points on the axis of y through which these lines pass :

(i) $y+3x=4$; (ii) $3y=x-4$; (iii) $2x+3y=9$;
(iv) $x+2y+5=0$; (v) $x+y=0$.

4. Write down the equations to the straight lines whose inclination θ to the x-axis and intercept c on the y-axis are :

(i) $\theta=30^\circ, c=1$; (ii) $\theta=135^\circ, c=-5$; (iii) $\theta=60^\circ, c=2$;
(iv) $\theta=45^\circ, c=\frac{1}{2}$, (v) $\theta=\tan^{-1}\frac{3}{4}, c=\frac{1}{4}$.

5. Find the intercepts on the axes of the following lines :

(i) $2x+3y=6$; (ii) $4x+5y+20=0$; (iii) $3x-5y-9=0$;
(iv) $5x-9y+13=0$; (v) $7x-5y=1$.

6. Find the equations to the straight lines cutting off intercepts a and b from the axes of x and y respectively where

(i) $a=2, b=3$; (ii) $a=\frac{7}{4}, b=-\frac{5}{2}$; (iii) $a=-3, b=\frac{2}{3}$;
(iv) $a=-\frac{3}{2}, b=-4$.

7. Reduce to the perpendicular form the following equations :

(i) $\sqrt{3}x+y=4$; (ii) $x-y+5\sqrt{2}=0$; (iii) $x+\sqrt{3}y+14=0$;
(iv) $x+y+2=0$.

8. Find the lengths of the perpendiculars from the origin on the lines whose equations are :

(i) $3x+4y-5=0$; (ii) $12x-5y-26=0$ and
(iii) $2x+3y+4=0$.

9. Trace the straight lines given by the following equations :

(i) $x=-3$; (ii) $2x=3y$; (iii) $3x+4y+12=0$;
(iv) $x\sqrt{3}-y-2=0$.

10. Find the equation to the straight line whose gradient is m and which cuts off an intercept b from the axis of x.

11. A straight line passes through the point $(-2, 3)$ and has a gradient $\frac{1}{3}$; find its equation. Find also the intercepts cut off by the line from the axes.

12. Find the equation to the straight line which is inclined at an angle 45° to the axis of x and which bisects the join of the points $(4, 7)$ and $(6, 5)$.

13. Find the equation to the straight line which passes through the point $(5, 3)$ and cuts off equal positive intercepts from the axes.

14. Find the equation to the straight line which passes through the point $(4, -7)$ and makes intercepts on the axes

(i) equal in magnitude and of the same sign ;

(ii) equal in magnitude but opposite in sign.

15. Find the equation to the straight line which passes through the point $(5, 6)$ and has intercepts on the axes equal in magnitude but opposite in sign. Find also the coordinates of the point at which the ordinate is double the abscissa. [*C. U.*]

16. Find the equation to the straight line which passes through the point (a, b) and is such that the portion of it intercepted between the axes is bisected at this point.

17. A straight line passes through the point $(4, 5)$ and is such that the portion of it intercepted between the axes is divided at the point in the ratio $4 : 5$. Find its equation.

18. Find the equation to the straight line which passes through the point $(2, 3)$ and is such that the sum of its intercepts on the axes is 10.

19. A straight line forms a right-angled triangle with the axes of coordinates. If the hypotenuse is 13 and the area of the triangle is 30, find the equation of the straight line. [*C. U.*]

20. Find the equation to the straight line passing through the point $(2, 3)$ and parallel to the line joining the points $(4, -7)$ and $(-7, 4)$.

21. Find the equation to the straight line passing through the point $(-3, 4)$ and parallel to the straight line $y+3=0$.

22. Find the equation to the straight line which passes through the point P $(4, 3)$ and is parallel to the line $5x-12y+7=0$; also determine the length intercepted on this line between the point P and the straight line $x+y=24$.

23. A straight line cuts off intercepts 7 and $5\frac{1}{4}$ from the axes ; find its equation and determine the ratio in which the join of the points $(-9, 5)$ and $(7, 9)$ is divided by this line.

24. Find the equations to the straight lines passing through the following pairs of points :

(i) $(0, 0)$ and $(3, 5)$; (ii) $(4, -9)$ and $(-3, 7)$;

(iii) $(1, 2)$ and $(-1, -2)$; (iv) $(0, a)$ and $(-b, 0)$;

(v) $(at_1^2, 2a'_1)$ and $(at_2^2, 2at_2)$; (vi) $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$;

(vii) $(a \cos \phi_1, b \sin \phi_1)$ and $(a \cos \phi_2, b \sin \phi_2)$.

(viii) $(a \sec \phi_1, b \tan \phi_1)$ and $(a \sec \phi_2, b \tan \phi_2)$.

25. Find the equations to the sides of the triangle whose vertices are

(i) $(0, 0)$, $(5, -2)$ and $(6, 9)$;

(ii) $(-4, 3)$, $(7, -3)$ and $(5, 8)$.

26. Find the equation to the straight line passing through the point (1, 2) and the mid-point of the join of (3, −4) and (5, −6).

27. Find the equations to the straight lines which pass through the origin and trisect the portion of the straight line $4x+3y-12=0$ intercepted between the axes.

28. If the points (a, b), (a', b'), $(a-a', b-b')$ are collinear, show that their join passes through the origin and that $ab'=a'b$. [*C. U.*]

29. Verify that the three points (1, 5), (3, 14) and (−1, −4) are collinear. Also find the line of collinearity. [*C. U.* 1957]

30. What does the equation $Ax+By+C=0$ represent

(*i*) for different values of C, A and B remaining constant ;

(*ii*) for different values of A, B and C remaining constant ?

31. Find the equations of the tangents to the circle whose centre is at the origin and radius 2, at the extremities of a diameter making an angle 30° with the axis of x.

32. Show that the distance of the point (x_0, y_0) from the line $ax+by+c=0$, measured parallel to a line making an angle θ with the x-axis, is [*C. U.* 1954]

$$-\frac{ax_0+by_0+c}{a\cos\theta+b\sin\theta}.$$

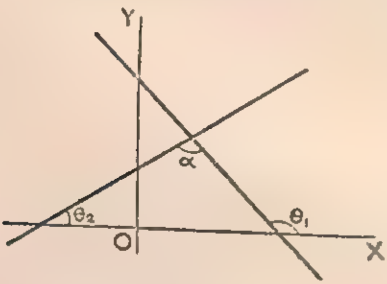
Answers :

1. (*i*) 1 ; (*ii*) 7 ; (*iii*) −3 ; (*iv*) $\frac{9}{5}$; (*v*) 0.
2. (*i*) 45° ; (*ii*) 135° ; (*iii*) 0 ; (*iv*) 90°.
3. (*i*) −3, (0, 4) ; (*ii*) $\frac{1}{3}$, $(0, -\frac{4}{3})$; (*iii*) $-\frac{2}{5}$, (0, 3) ;
(*iv*) $-\frac{1}{2}$, $(0, -\frac{5}{6})$; (*v*) −1, (0, 0).
4. (*i*) $x-\sqrt{3}y+\sqrt{3}=0$; (*ii*) $x+y+5=0$; (*iii*) $\sqrt{3}x-y+2=0$;
(*iv*) $2x-2y+1=0$; (*v*) $3x-4y+1=0$.
5. (*i*) 3, 2 ; (*ii*) −5, −4 ; (*iii*) 3, $-\frac{9}{5}$; (*iv*) $-\frac{13}{5}$, $\frac{13}{6}$;
(*v*) $\frac{1}{7}$, $-\frac{1}{5}$.
6. (*i*) $3x+2y=6$; (*ii*) $10x-14y=35$; (*iii*) $2x-9y+6=0$;
(*iv*) $16x+3y+12=0$.
7. (*i*) $x\cos 30°+y\sin 30°=2$; (*ii*) $x\cos 135°+y\sin 135°=5$;
(*iii*) $x\cos 240°+y\sin 240°=7$; (*iv*) $x\cos 225°+y\sin 225°=\sqrt{2}$.
8. (*i*) 1 ; (*ii*) 2 ; (*iii*) $\frac{4}{\sqrt{13}}$
10. $y=m(x-b)$.
11. $x-2y+8=0$; −8 and 4.
12. $x-y+1=0$
13. $x+y=8$.
14. (*i*) $x+y+3=0$; (*ii*) $x-y=11$.
15. $x-y+1=0$; (1, 2).
16. $\frac{x}{a}+\frac{y}{b}=2$.
17. $25x+16y=180$.
18. $3x+2y-12=0$ or, $x+y=5$.

19. $12x+5y=\pm 60,\ 5x+12y=\pm 60.$ 20. $x+y=5.$ 21. $y=4.$
22. $5x-12y+16=0$; 13. 23. $3x+4y=21$; 7 : 9.
24. (i) $3y-5x=0$; (ii) $16x+7y-1=0$; (iii) $2x-y=0$;
(iv) $ax-by+ab=0$; (v) $y(t_1+t_2)-2x=2at_1t_2$;
(vi) $x+yt_1t_2=c(t_1+t_2)$;
(vii) $\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}$;
(viii) $\frac{x}{a}\cos\frac{\phi_1-\phi_2}{2}-\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1+\phi_2}{2}$.
25. (i) $2x+5y=0,\ 11x-y-57=0,\ 3x-2y=0.$
(ii) $6x+11y-9=0,\ 11x+2y-71=0.\ 5x-9y+47=0.$
26. $7x+3y-13=0.$ 27. $2x-3y=0,\ 8x-3y=0.$ 29. $9x-2y+1=0.$
30. (i) A system of parallel straight lines having gradient $-\frac{A}{B}$;
(ii) a system of concurrent lines passing through the point $\left(0,\ -\frac{C}{B}\right)$.
31. $\sqrt{3}x+y=\pm 4.$

IV-9. Angle between two given lines :

(i) Let the equations of the given lines be



$y=m_1x+c_1$ and $y=m_2x+c_2$ and let α be the angle between them.

If θ_1 and θ_2 be the inclinations of the lines to the x-axis,

then $\tan\theta_1=m_1$

and $\tan\theta_2=m_2$.

Now, from the figure, $\alpha=\theta_1-\theta_2$.

$$\therefore\ \tan\alpha=\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

$$i.e.,\ \tan\alpha=\frac{m_1-m_2}{1+m_1m_2} \quad \dots \quad \dots \quad (1)$$

giving the angle between the two given lines.

Note : When two lines (not at right angles) intersect, of the two angles formed, one is acute and the other is obtuse. Hence, in any particular problem if the expression for $\tan\alpha$ is found to be positive, α is the acute angle between the lines and if negative it is the obtuse angle.

(*ii*) If the equations of the lines be

$$A_1x+B_1y+C_1=0$$

and $$A_2x+B_2y+C_2=0,$$

writing them in the gradient form, we have

$$m_1=-\frac{A_1}{B_1} \text{ and } m_2=-\frac{A_2}{B_2}$$

where m_1 and m_2 are the respective gradients of the lines.

$$\text{Hence, } \tan\alpha=\frac{m_1-m_2}{1+m_1m_2}=\frac{-\frac{A_1}{B_1}+\frac{A_2}{B_2}}{1+\frac{A_1}{B_1}\frac{A_2}{B_2}}$$

i.e., $$\tan\alpha=\frac{\mathbf{B_1A_2-A_1B_2}}{\mathbf{A_1A_2+B_1B_2}} \qquad \ldots \qquad \ldots \qquad (2)$$

(*iii*) If the equations be given in the form

$$x\cos\alpha_1+y\sin\alpha_1=p_1$$

and $$x\cos\alpha_2+y\sin\alpha_2=p_2$$

α_1, α_2 are the angles which the perpendiculars from the origin to the lines make with the axis of x. Hence, clearly

$$\alpha=(\alpha_1-\alpha_2) \text{ or, } \pi-(\alpha_1-\alpha_2).$$

IV-10. Condition of parallelism and perpendicularity of two lines :

(1) If the **lines are parallel**, $\theta_1=\theta_2$

$$\therefore \quad \mathbf{m_1=m_2} \text{ or, } \frac{\mathbf{A_1}}{\mathbf{A_2}}=\frac{\mathbf{B_1}}{\mathbf{B_2}}.$$

(2) If the **lines are at right angles**,

$\alpha=90^\circ$, hence, $\tan\alpha=\infty$, which gives

$$\mathbf{m_1m_2=-1} \text{ or, } \mathbf{A_1A_2+B_1B_2=0}.$$

The condition of perpendicularity may also be derived independently thus :

From the figure of the previous article, if

$$\alpha=90^\circ,\ \theta_1=90^\circ+\theta_2.$$

$$\therefore \quad \tan\theta_1=\tan(90^\circ+\theta_2)=-\cot\theta_2=-\frac{1}{\tan\theta_2}.$$

Hence, $\tan\theta_1.\tan\theta_2=-1$, *i.e.*, $m_1m_2=-1$.

IV-11. Working rule :

The straight line $A_2x+B_2y+C_2=0$ is parallel to $A_1x+B_1y+C_1=0$ if $\frac{A_2}{A_1}=\frac{B_2}{B_1}=k$ (say) *i.e.*, $A_2=A_1k$, $B_2=B_1k$. Hence, the equation to a line parallel to $A_1x+B_1y+C_1=0$ is $A_1kx+B_1ky+C_2=0$, *i.e.*, $A_1x+B_1y+C'=0$ where $C'=\frac{C_2}{k}$ (constant).

We, therefore, see that the equations of two parallel lines differ only in the constant term.

Again, if we consider the straight lines,

$$A_1x+B_1y+C_1=0$$

and

$$B_1x-A_1y+C_2=0$$

whose gradients m_1 and m_2 are respectively

$$-\frac{A_1}{B_1} \text{ and } \frac{B_1}{A_1}, \text{ we find}$$

$$m_1m_2=\left(-\frac{A_1}{B_1}\right)\left(\frac{B_1}{A_1}\right)=-1$$

showing that the lines are at right angles.

If we examine the two equations, we find that the second equation has been formed from the first by interchanging the coefficients of x and y and changing the sign of one of them and adding a different constant. Hence, the following **working rule :**

(1) *To write down the equation to a straight line parallel to a given straight line, simply change the constant term, and*

(2) *to write down the equation to a straight line perpendicular to a given straight line, interchange the coefficients of x and y, change the sign of one of them and add a different constant.*

Example. Given a straight line whose equation is $2x+3y+4=0$;

any line parallel to it is $2x+3y+k=0$; and any line perpendicular to it is $3x-2y+k'=0$.

IV-12. Point of intersection of two straight lines :

Let the equations of the straight lines be

$$a_1x+b_1y+c_1=0$$

and $$a_2x+b_2y+c_2=0.$$

Now the point of intersection of the two lines is the only point which lies on both the lines and therefore its coordinates must satisfy both the equations.

If, therefore, (α, β) be this point of intersection, we have

$$a_1\alpha+b_1\beta+c_1=0$$

$$a_2\alpha+b_2\beta+c_2=0$$

Hence, by the rule of cross multiplication,

$$\frac{\alpha}{b_1c_2-b_2c_1}=\frac{\beta}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1};$$

so that the coordinates of the required point of intersection are

$$\alpha=\frac{\mathbf{b_1c_2-b_2c_1}}{\mathbf{a_1b_2-a_2b_1}} \text{ and } \beta=\frac{\mathbf{c_1a_2-c_2a_1}}{\mathbf{a_1b_2-a_2b_1}}.$$

Remark. We get finite values of the coordinates only if $a_1b_2-a_2b_1\neq 0$. If however, $a_1b_2-a_2b_1=0$, *i.e.*, $\frac{a_1}{a_2}=\frac{b_1}{b_2}$, then both the coordinates are infinitely large ; but if this condition holds, we know from Art. IV-10(i) that the lines are parallel. Hence, two parallel lines must be looked upon as meeting at a point which is situated at an infinite distance from the origin.

IV-13. Condition of concurrence of three lines :

Let the equations of the lines be

$$a_1x+b_1y+c_1=0 \quad \cdots \quad \cdots \quad (1)$$

$$a_2x+b_2y+c_2=0 \quad \cdots \quad \cdots \quad (2)$$

and $$a_3x+b_3y+c_3=0 \quad \cdots \quad \cdots \quad (3)$$

If the three lines are concurrent, the point of intersection of any two of them must lie on the third. Now, the coordinates of the point of intersection of (2) and (3) obtained by solving these equations, are

$$\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2} \text{ and } \frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}.$$

If this point lies on (1) we must have

$$a_1\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}\right)+b_1\left(\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right)+c_1=0$$

i.e., $\mathbf{a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0}$ which is the required condition of concurrence of the three given lines.

Another form :

If three constants p, q, r (not zero) can be found so that

$$\mathbf{p(a_1x+b_1y+c_1)+q(a_2x+b_2y+c_2)} \\ \mathbf{+r(a_3x+b_3y+c_3)=0} \textit{ identically}$$

(*i.e.*, the expression should vanish irrespective of the values of x and y, which requires that the coefficients of x and y and the absolute term should separately vanish)
then the three lines must meet in a point.

Let (α, β) be the point of intersection of (2) and (3).

$$\begin{array}{ll}\text{Then} & a_2\alpha+b_2\beta+c_2=0 \\ \text{and} & a_3\alpha+b_3\beta+c_3=0\end{array}\Bigg\} \quad \cdots \quad \cdots \quad (A)$$

But $p(a_1\alpha+b_1\beta+c_1)+q(a_2\alpha+b_2\beta+c_2)+r(a_3\alpha+b_3\beta+c_3)=0$ since the expression vanishes for all values (and therefore also for the values α and β) of x and y.

$$\therefore \quad a_1\alpha+b_1\beta+c_1=-\frac{q}{p}(a_2\alpha+b_2\beta+c_2)-\frac{r}{p}(a_3\alpha+b_3\beta+c_3)$$

$$=-\frac{q}{p}\times 0-\frac{r}{p}\times 0 \cdots \quad \textit{from } (A)$$

$$=0.$$

Hence, (α, β) is also a point on the line (1), that is, the three given lines are concurrent.

Note : The latter form of the condition may be applied with advantage when the form of the equations would at once suggest the values of p, q and r, and in most cases these values would be ound to be each unity, when by simply adding the left-hand members of the equations the sum would be found to vanish identically.

In numerical examples, however, the student is advised to find the coordinates of the point of intersection of any two of the lines and then

to substitute these in the third equation to see whether or not the equation is satisfied.

IV-14. Lines through the intersection of two given lines :

The equation

$$\mathbf{a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0} \quad \cdots \quad (A)$$

where λ is an arbitrary constant, represents **any** *straight line passing through the point of intersection of the lines*

$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0.$$

Let (α, β) be the point of intersection of the lines

$a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, so that

$$\left.\begin{array}{l} a_1\alpha+b_1\beta+c_1=0 \\ \text{and} \quad a_2\alpha+b_2\beta+c_2=0 \end{array}\right\} \quad \cdots \quad \cdots \quad (B)$$

Now (α,β) satisfies the equation (A) ; for on substitution

$$a_1\alpha+b_1\beta+c_1+\lambda\,(a_2\alpha+b_2\beta+c_2)$$
$$=0+\lambda.\,0 \quad \cdots \quad \textit{from } (B)$$
$$=0,$$

so that equation (A) represents some locus through (α, β).

Also the equation being one of first degree in x and y the locus represented by (A) is a straight line.

Further, by properly choosing the arbitrary constant λ, the straight line represented by the equation (A) may be made to satisfy any other condition.

Hence, the equation (A) represents *any* straight line through (α, β).

Conversely,

A straight line whose equation is of the form

$$\mathbf{a_1x+b_1y+c_1+\lambda\,(a_2x+b_2y+c_2)=0} \quad \cdots \quad (A)$$

where λ is an arbitrary constant, **always passes through one fixed point** *whatever be the value of λ.*

Consider, $a_1x+b_1y+c_1=0 \quad \cdots \quad \cdots \quad (1)$

$a_2x+b_2y+c_2=0 \quad \cdots \quad \cdots \quad (2)$

Since the equations do not involve λ, they represent two fixed lines.

Their point of intersection is therefore a fixed point.

Since the coordinates of this point of intersection satisfy equations (1) and (2) they also satisfy equation (A).

Hence, whatever may be the value of λ the straight line (A) always passes through a fixed point, *viz.*, the point of intersection of the two fixed lines (1) and (2).

WORKED OUT EXAMPLES

Ex. 1. *Find the equation to the straight line which passes through the point* $(-3, 4)$ *and is*

(*i*) *parallel to the straight line* $2x+3y+4=0$;

(*ii*) *perpendicular to the straight line* $2x+3y+4=0$.

(*i*) ***First method*** :

Any straight line parallel to $2x+3y+4=0$ is given by

$$2x+3y+k=0 \qquad \ldots \qquad \ldots \qquad (1)$$

The line (1) passes through $(-3, 4)$ if

$$2(-3)+3.4+k=0$$

i.e., if $k=-6$.

$\therefore$ The required equation is

$$2x+3y-6=0.$$

[In this method we first take any line parallel to the given line and then choose that one of the system which passes through the given point. Equation (1) represents different lines for different values of k but all parallel to the given line. Of these lines there is one which also passes through the given point $(-3, 4)$ and $k=-6$ corresponds to this particular line.]

Second method :

The equation to the given line can be written in the form

$$y=-\frac{2}{3}x-\frac{4}{3}. \qquad \ldots \qquad \ldots \qquad (1)$$

The gradient of the line is therefore $-\frac{2}{3}$. Any line passing through the point $(-3, 4)$ is given by

$$y-4=m(x+3). \quad \cdots \quad \cdots \quad (2)$$

The line (2) is parallel to (1) if $m=-\frac{2}{3}$.

Hence, the required equation is

$$y-4=-\tfrac{2}{3}(x+3)$$

i.e., $2x+3y-6=0$.

[In this method we first take any line passing through the given point and then choose that one of the system which is parallel to the given line. Equation (2) represents different lines for different values of m but all passing through the point $(-3, 4)$. Of these lines there is one which is also parallel to the given line and $m=-\frac{2}{3}$ corresponds to this particular line.]

Third method :

Let the equation to the required line be

$$y=mx+c. \quad \cdots \quad \cdots \quad (1)$$

Since (1) passes through $(-3, 4)$ we have

$$4=-3m+c \quad \cdots \quad \cdots \quad (2)$$

Also the line (1) being parallel to

$$2x+3y+4=0$$

i.e., to $y=-\frac{2}{3}x-\frac{4}{3}$, we must have

$$m=-\tfrac{2}{3}. \quad \cdots \quad \cdots \quad (3)$$

Hence, from (2) and (3), we get $c=2$.

On substitution for m and c in (1), the required equation to the line is found to be

$$y=-\tfrac{2}{3}x+2$$

i.e., $2x+3y-6=0$.

[In this method we assume the equation of the required line and then find the constants of the equation from the given conditions.]

(*ii*) Any line perpendicular to $2x+3y+4=0$ is given by

$$3x-2y+k=0 \quad \cdots \quad \cdots \quad (1)$$

The line (1) passes through $(-3, 4)$ if

$$3(-3)-2.4+k=0$$

i.e., if $k=17$.

The required equation is therefore

$$3x-2y+17=0.$$

Ex. 2. *Find the value of k for which the three lines $2x-3y+k=0$, $3x-4y-1=0$ and $4x-5y-2=0$ may be concurrent.*

Solving the equations

$$3x-4y-1=0$$

and $$4x-5y-2=0$$

we get $x=3$ and $y=2$, which are therefore the coordinates of the point of intersection of these lines. If the three given lines are concurrent, the point $(3, 2)$ must lie on the line $2x-3y+k=0$, which requires

$$2\times 3-3\times 2+k=0$$

i.e., $$k=0$$

which gives the required value.

Ex. 3. *Prove that the perpendicular bisectors of the sides of any triangle are concurrent.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle.

If D be the mid-point of BC, then its coordinates are

$$\frac{x_2+x_3}{2} \text{ and } \frac{y_2+y_3}{2}.$$

Also the gradient of the line BC is $\dfrac{y_2-y_3}{x_2-x_3}$.

Hence, the equation to the line through D perpendicular to BC is

$$y-\frac{y_2+y_3}{2}=-\frac{x_2-x_3}{y_2-y_3}\left(x-\frac{x_2+x_3}{2}\right)$$

[since the gradient m of this line is given by

$$m \times \frac{y_2 - y_3}{x_2 - x_3} = -1 \quad i.e., \quad m = -\frac{x_2 - x_3}{y_2 - y_3} \Big]$$

which reduces to

$$2x(x_2 - x_3) + 2y(y_2 - y_3) - (x_2^2 - x_3^2) - (y_2^2 - y_3^2) = 0 \ldots (1)$$

Similarly, the lines through the mid-points of CA and AB perpendiculars respectively to these sides are

$$2x(x_3 - x_1) + 2y(y_3 - y_1) - (x_3^2 - x_1^2) - (y_3^2 - y_1^2) = 0 \quad \cdots \quad (2)$$

$$2x(x_1 - x_2) + 2y(y_1 - y_2) - (x_1^2 - x_2^2) - (y_1^2 - y_2^2) = 0 \quad \cdots \quad (3)$$

The left-hand members of the equations (1), (2) and (3) when added together vanish identically (*i.e.*, we choose $p = q = r = 1$ in Art. IV-13). Hence, the three lines are concurrent.

Ex. 4. *Find the equation of the straight line passing through the origin and through the point of intersection of the lines* $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Any line through the intersection of the two given lines is

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0 \quad \ldots \quad \ldots \quad (1)$$

The line (1) passes through the origin (0, 0) if

$$a_1.0 + b_1.0 + c_1 + \lambda(a_2.0 + b_2.0 + c_2) = 0$$

i.e., if $\lambda = -\dfrac{c_1}{c_2}$.

$\therefore$ Substituting for λ in (1), the required equation is

$$a_1x + b_1y + c_1 - \frac{c_1}{c_2}(a_2x + b_2y + c_2) = 0.$$

i.e., $$c_2(a_1x + b_1y) - c_1(a_2x + b_2y) = 0.$$

Ex. 5. *If* a *and* b *in the equation* $ax + by + c = 0$ *vary subject to the condition that* $a + b$ *is constant, prove that the line represented by the equation always passes through a fixed point.*

Let $a + b = k$ (constant)

so that $b = k - a$

The equation $ax+by+c=0$ can be written as

$$ax+(k-a)y+c=0$$

i.e., $$a(x-y)+ky+c=0$$

i.e., $$(x-y)+\frac{k}{a}\left(y+\frac{c}{k}\right)=0$$

which, for different values of a, represents different lines but all passing through a fixed point, *viz.*, the point of intersection of the two fixed straight lines

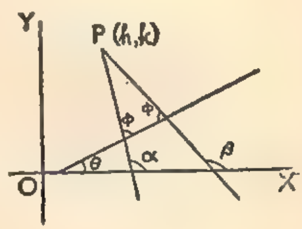
$$x-y=0 \quad \text{and} \quad y+\frac{c}{k}=0 \qquad \text{(Art. IV-14)}$$

i.e., the fixed point $\left(-\frac{c}{k}, -\frac{c}{k}\right)$.

Ex. 6. *Find the equations to the straight lines which pass through the point (h, k) and make angles ϕ with the straight line $ax+by+c=0$.*

Let θ be the angle which the line $ax+by+c=0$ makes with the x-axis, so that

$$\tan\theta = \text{gradient of the line} = -\frac{a}{b}.$$



If α and β be the angles which the required lines make with the x-axis, then the equations are

$$y-k=\tan\alpha(x-h) \qquad \dots (1)$$

and $$y-k=\tan\beta(x-h) \qquad \dots (2)$$

Now from the figure

$$\tan\alpha=\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{-\frac{a}{b}+\tan\phi}{1+\frac{a}{b}\tan\phi}$$

$$=\frac{b\tan\phi-a}{a\tan\phi+b}$$

$$\tan\beta=\tan\{(\pi-\phi)+\theta\}=\tan(\pi+\theta-\phi)=\tan(\theta-\phi)$$

$$=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{-\frac{a}{b}-\tan\phi}{1-\frac{a}{b}\tan\phi}=\frac{b\tan\phi+a}{a\tan\phi-b}$$

Hence, the required equations are

$$y-k=\frac{b\tan\phi-a}{a\tan\phi+b}(x-h)$$

and $$y-k=\frac{b\tan\phi+a}{a\tan\phi-b}(x-h).$$

[Except when $\phi=\frac{\pi}{2}$ or 0, there will be two such straight lines satisfying the given conditions.]

EXERCISE IV(B)

1. Find the angles between the following pairs of straight lines :

   (i) $\sqrt{3}x-y=0$ and $x-\sqrt{3}y+3=0$;

   (ii) $y=x+2$ and $y=2$;

   (iii) $3x-4y+5=0$ and $x+7y-9=0$;

   (iv) $4x-7y+3=0$ and $14x+8y+5=0$;

   (v) $5x-y+1=0$ and $x-3y+2=0$.

2. Find the angle between the lines

   (i) $x\cos 15^\circ-y\sin 15^\circ+5=0$, and
   $x\sin 105^\circ+y\cos 105^\circ-5=0$.

   (ii) $x\cos 25^\circ+y\sin 25^\circ-7=0$, and
   $x\sin 25^\circ+y\cos 25^\circ+7=0$.

3. Prove that the lines $4x+6y+5=0$ and $6x+9y-7=0$ are parallel.

4. Prove that the lines $3x-4y+7=0$ and $8x+6y-9=0$ are at right angles.

5. Prove analytically that the line joining the middle points of any two sides of a triangle is parallel to the third side.

6. Write down the equation to the straight line passing through the origin and

   (i) parallel to the line $lx+my+n=0$;

   (ii) perpendicular to the line $lx+my+n=0$.

7. Find the equation to the straight line

   (i) which passes through the point $(5, -3)$ and is parallel to the line $7x+9y-11=0$;

   (ii) which passes through the point $(-4, 7)$ and is perpendicular to the line $5x-7y+2=0$;

   (iii) which passes through the point $(2, 3)$ and is perpendicular to the line joining the points $(3, -4)$ and $(-5, 6)$.

8. Find the equation to the straight line passing through the point (x_1, y_1) and

(i) parallel to the straight line $lx+my+n=0$;

(ii) perpendicular to the straight line $lx+my+n=0$.

9. Find the equation of the straight line which passes through (x_1, y_1) and is perpendicular to the join of (x_2, y_2) and (x_3, y_3). [C. U.]

10. Derive the Cartesian equation of the perpendicular bisector of the line joining the two points (x_1, y_1) and (x_2, y_2). [C. U. 1956].

11. Find the coordinates of the points of intersection of the following pairs of straight lines :

(i) $2x-3y+11=0$ and $3x-4y+13=0$;

(ii) $7x+2y=0$ and $5x+8y+23=0$;

(iii) $y=m_1x+\dfrac{a}{m_1}$ and $y=m_2x+\dfrac{a}{m_2}$;

(iv) $\dfrac{x}{a}+\dfrac{y}{b}=1$ and $\dfrac{x}{b}+\dfrac{y}{a}=1$. [C. U.]

12. Show that the area of the triangle formed by the straight lines whose equations are

$$y=m_1x+c_1,\ y=m_2x+c_2 \text{ and } x=0$$

is $\dfrac{1}{2}\dfrac{(c_1-c_2)^2}{m_2-m_1}$. [C. U. 1955]

13. Prove that the area of the triangle formed by the straight lines whose equations are

$$y=m_1x+c_1,\ y=m_2x+c_2 \text{ and } y=m_3x+c_3$$

is $\dfrac{1}{2}\left\{\dfrac{(c_2-c_3)^2}{m_2-m_3}+\dfrac{(c_3-c_1)^2}{m_3-m_1}+\dfrac{(c_1-c_2)^2}{m_1-m_2}\right\}$.

14. Verify that the three lines $y=2$, $y-\sqrt{3}x=5$, $y+\sqrt{3}x=4$ form an equilateral triangle. Further compute the area of this triangle. [C. U. 1957]

15. Show that the lines

$$(a+b)\,x+(a-b)\,y-2ab=0,$$

$$(a-b)\,x+(a+b)\,y-2ab=0 \text{ and } x+y=0$$

form an isosceles triangle whose vertical angle is $2\tan^{-1}\dfrac{a}{b}$.

Determine the coordinates of its centroid. [C. U.]

16. Find the coordinates of the foot of the perpendicular

(i) from the point (2, 3) on the line $x+y-11=0$; [C. U.]

(ii) from the point (α, β) on the line $Ax+By+C=0$.

17. Prove that the following sets of three lines are concurrent and find the respective points of concurrence :

(i) $4x-3y-31=0$, $7x-5y-56=0$, $11x-9y-80=0$;

(*ii*) $2x-3y-5=0,\ 10x+18y+19=0,\ 14x-12y-23=0$;

(*iii*) $\dfrac{x}{a}+\dfrac{y}{b}=1, \dfrac{x}{q}+\dfrac{y}{a}=1,\ x=y$;

(*iv*) $ax+(b+c)y=p,\ \ bx+(c+a)\,y=p,\ \ cx+(a+b)\,y=p$.

18. Find the value of k for which the three lines

$$7x-11y+3=0,\ 4x+3y-9=0 \text{ and } 13x+ky-48=0$$

are concurrent.

19. Verify that the three lines

$$x-y-7=0,\ x+2y+6=0 \text{ and } 2x+y-1=0$$

pass through a common point and that this point is equidistant from the three points $(5,\ -4)$, $(3,\ -2)$ and $(1, -6)$. [*C. U.* 1956]

20. Given $l+m+n=0$; prove that the three lines given by the equations $lx+my+n=0$, $mx+ny+l=0$ and $nx+ly+m=0$ are concurrent.

21. In any triangle, prove that

(*i*) the medians are concurrent,

(*ii*) the perpendiculars from the vertices on the opposite sides are concurrent.

22. Find the ortho-centre of the triangle

(*i*) whose sides are
$x-y+1=0$, $3x+y-17=0$ and $x+5y+13=0$;

(*ii*) whose vertices are
$(1,\ 5)$, $(7,\ 2)$ and $(4,\ 9)$.

23. Show that the ortho-centre of the triangle formed by the straight lines whose equations are

$$y=m_1x+\frac{a}{m_1},\ y=m_2x+\frac{a}{m_2} \text{ and } y=m_3x+\frac{a}{m_3}$$

is the point $\left\{-a,\ a\left(\dfrac{1}{m_1}+\dfrac{1}{m_2}+\dfrac{1}{m_3}+\dfrac{1}{m_1m_2m_3}\right)\right\}$.

24. For all values of the parameter λ, prove that the line $5x+y-11-\lambda(2x-7y+3)=0$ goes through a certain fixed point and determine its coordinates. [*C. U.* 1950]

25. Prove that the lines

$$a_1x+b_1y+c_1=0,\ \ a_2x+b_2y+c_2=0$$
$$\text{and}\quad (pa_1+qa_2)x+(pb_1+qb_2)y+(pc_1+qc_2)=0$$

pass through a common point and find the coordinates of this point.

26. Find the equation to the straight line passing through

(*i*) the origin and the point of intersection of the lines
$5x-9y+20=0$ and $3x+7y+10=0$;

(ii) the point $(-4, 7)$ and the point of intersection of the lines $3x+4y-5=0$ and $4x-5y+7=0$;

(iii) the intersection of the lines
$$7x+13y-11=0 \text{ and } 5x-9y+13=0$$
and tne intersection of the lines
$$2x-5y-25=0 \text{ and } 7x+3y-26=0.$$

27. Find the equation of the line passir g through the point $(3, 2)$ and the intersection of the lines $3x+y-5=0$ and $x+5y+3=0$. Find also the area of the triangle cut off from the coordinate axes by the line.

[*C. U.*]

28. Find the equation of the straight line

(i) passing through the intersection of the lines $x+4y+9=0$ and $5x+y-12=0$, and parallel to the line $2x-3y+5=0$;

(ii) passing through the intersection of the lines $2x-3y+4=0$ and $3x+4y-5=0$, and perpendicular to the line $6x-7y+8=0$.

[*C. U.*]

29. Find the equation to the straight line which passes through the intersection of the lines $5x-6y+7=0$ and $3x+7y-9=0$ and cuts off equal intercepts from the axes.

30. Find the equation to the straight line passing through the point of intersection of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, and

(i) parallel to the x-axis ;

(ii) parallel to the y-axis.

31. Find the equation of the straight line which passes through the point of intersection of two given straight lines $ax+by+c=0$ and $a'x+b'y+c'=0$ and through a given point (h, k). [*C. U.*]

32. Find the equation to the straight line passing through the intersection of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ and

(i) parallel to the line $a_3x+b_3y+c_3=0$;

(ii) perpendicular to the line $a_3x+b_3y+c_3=0$.

33. A straight line moves so that the sum of the reciprocals of its intercepts on the axes is constant. Show that it passes through a fixed point.

[*C. U.*]

34. If $lx+my+n=0$, where l, m, n are not constants, is the equation of a variable straight line and l, m, n are connected by the relation $al+bm+cn=0$, where a, b, c are constants, show that the variable line passes through a fixed point. [*C. U.* 1953]

35. Prove that the diagonals of the parallelogram formed by the four straight lines

$\sqrt{3}x+y=0$, $\sqrt{3}y+x=0$, $\sqrt{3}x+y=1$ and $\sqrt{3}y+x=1$ are at right angles to one another. [*C. U.* 1953]

36. Prove that the diagonals of the parallelogram formed by the four straight lines $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{b}+\frac{y}{a}=1$, $\frac{x}{a}+\frac{y}{b}=2$ and $\frac{x}{b}+\frac{y}{a}=2$ are at right angles to one another. [*C. U.*]

37. Find the coordinates of the middle points of the three diagonals of the complete quadrilateral formed by the lines $x=0$, $y=0$, $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{a'}+\frac{y}{b'}=1$, and show that these three points are collinear. [*C. U.*]

38. Find the equations to the straight lines which pass through the point (3, 2) and are inclined at an angle 45° to the straight line $x-2y=4$.

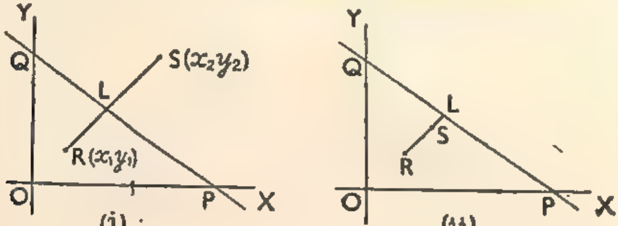
39. Find the equations of the straight lines passing through the point (7, 9) and inclined at an angle 60° to the straight line $x-\sqrt{3}y-2\sqrt{3}=0$.

Answers :

1. (*i*) 30° ; (*ii*) 45° ; (*iii*) 45° ; (*iv*) 90° ; (*v*) $\tan^{-1}\frac{17}{4}$. 2. (*i*) 0° ; (*ii*) 90°. 6. (*i*) $lx+my=0$; (*ii*) $mx-ly=0$. 7. (*i*) $7x+9y-8=0$; (*ii*) $7x+5y-7=0$; (*iii*) $4x-5y+7=0$. 8. (*i*) $l(x-x_1)+m(y-y_1)=0$; (*ii*) $m(x-x_1)-l(y-y_1)=0$. 9. $(x-x_1)(x_2-x_3)+(y-y_1)(y_2-y_3)=0$. 10. $2x(x_1-x_2)+2y(y_1-y_2)=x_1^2-x_2^2+y_1^2-y_2^2$. 11. (*i*) (5, 7) ; (*ii*) $(1, -\frac{7}{3})$; (*iii*) $\left\{\frac{a}{m_1m_2}, a\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right\}$; (*iv*) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$. 14. $\frac{24}{13}\sqrt{3}$. 15. $\left(\frac{b}{3}, \frac{b}{3}\right)$. 16. (*i*) (5, 6) ; (*ii*) $\left\{\frac{B(B\alpha-A\beta)-AC}{A^2+B^2}, \frac{A(A\beta-B\alpha)-BC}{A^2+B^2}\right\}$. 17. (*i*) (13, 7) ; (*ii*) $(\frac{1}{2}, -\frac{4}{3})$; (*iii*) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$; (*iv*) $\left(\frac{p}{a+b+c}, \frac{p}{a+b+c}\right)$. 18. $k=26$. 22. (*i*) (3, 0) ; (*ii*) $\frac{25}{11}, \frac{21}{11})$. 24. (2, 1). 25. $\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$. 26. (*i*) $x+23y=0$; (*ii*) $16x+11y-13=0$; (*iii*) $53x+71y-52=0$. 27. $3x-y-7=0$; $\frac{49}{5}$. 28. (*i*) $2x-3y-15=0$; (*ii*) $119x+102y-125=0$. 29. $53x+53y-71=0$. 30. (*i*) $(b_1a_2-b_2a_1)y+(c_1a_2-c_2a_1)=0$; (*ii*) $(a_1b_2-a_2b_1)x+c_1b_2-c_2b_1=0$. 31. $\frac{ax+by+c}{ah+bk+c}=\frac{a'x+b'y+c'}{a'h+b'k+c'}$. 32. (*i*) $\frac{a_1x+b_1y+c_1}{a_1b_3-a_3b_1}=\frac{a_2x+b_2y+c_2}{a_2b_3-a_3b_2}$; (*ii*) $\frac{a_1x+b_1y+c_1}{a_1a_3+b_1b_3}=\frac{a_2x+b_2y+c_2}{a_2a_3+b_2b_3}$. 38. $3x-y=7$, $x+3y=9$. 39. $x=7$, $x+\sqrt{3}y=7+9\sqrt{3}$.

IV-15. Position of points in relation to a given line :

Let the equation to the given line PQ be $Ax+By+C=0$, and let $R\ (x_1, y_1)$ and $S\ (x_2, y_2)$ be two given points.



(i) (ii)

Suppose RS joined meets PQ in L where $\frac{RL}{LS}=\frac{m}{n}$.

In fig. (i) where R and S are situated on opposite sides of PQ, the point L divides the join internally in the ratio $m:n$. Its coordinates are therefore $\frac{mx_2+nx_1}{m+n}$ and $\frac{my_2+ny_1}{m+n}$

Since this point lies on PQ, we have

$$A\frac{mx_2+nx_1}{m+n}+B\frac{my_2+ny_1}{m+n}+C=0$$

which gives

$$\frac{Ax_1+By_1+C}{Ax_2+By_2+C}=-\frac{m}{n}. \quad \cdots \quad \cdots \quad (1)$$

In fig. (ii) where R and S are situated on the same side of PQ, the point L divides the join externally in the ratio $m:n$. Its coordinates are therefore

$$\frac{mx_2-nx_1}{m-n} \text{ and } \frac{my_2-ny_1}{m-n}$$

and we easily have on substitution

$$\frac{Ax_1+By_1+C}{Ax_2+By_2+C}=+\frac{m}{n}. \quad \cdots \quad (2)$$

Since, $\frac{m}{n}$ is a positive quantity, the ratio $\frac{Ax_1+By_1+C}{Ax_2+By_2+C}$ is negative in (1) and positive in (2).

Hence, clearly, Ax_1+By_1+C and Ax_2+By_2+C must have opposite signs in the first case and same signs in the second. We therefore conclude—

The points (x_1, y_1) *and* (x_2, y_2) *are on the same side of the line* $Ax+By+C=0$ *if the expressions* Ax_1+By_1+C *and* Ax_2+By_2+C *have the same signs, and they are on opposite sides if the corresponding expressions have opposite signs.*

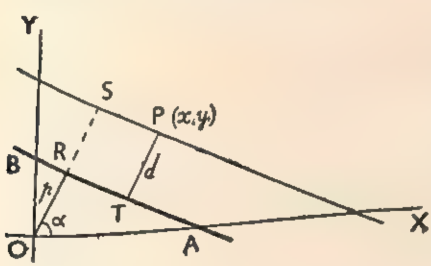
On substitution of the coordinates of the origin (0, 0) the expression becomes $A.0+B.0+C$, *i.e.*, C. Hence, the point (x_1, y_1) lies on the same side of the line in which the origin lies if Ax_1+By_1+C has the same sign as C and it lies on the other side if Ax_1+By_1+C has a sign opposite to that of C.

IV-16. Length of the perpendicular from a point to a line :

(1) Let the equation of the given line AB be taken in the form

$$x \cos \alpha + y \sin \alpha = p$$

where p is the length of the perpendicular OR from O on AB and the angle $XOR=\alpha$.



Let d be the length of the perpendicular PT from $P(x_1, y_1)$ to the line AB.

Draw through P a line parallel to AB to meet OR produced in S.

Since, $OS=OR+RS=p+d$, and $\angle XOS=\alpha$

the equation to the line PS is

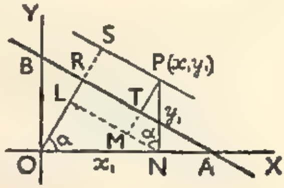
$$x \cos \alpha + y \sin \alpha = p+d.$$

Now point $P(x_1, y_1)$ lies on this line.

Hence,
$$x_1 \cos \alpha + y_1 \sin \alpha = p+d.$$

$$\therefore \quad \mathbf{d = x_1 \cos \alpha + y_1 \sin \alpha - p.}$$

Otherwise : Draw the ordinate PN and draw NL perpendicular to OS and PM perpendicular to NL.



We have

$$d = PT = RS$$
$$= OL + LS - OR$$
$$= ON \cos\alpha + NP \sin\alpha - OR$$

i.e., $d = x_1 \cos\alpha + y_1 \sin\alpha - p$.

(2) If the equation of the line AB be $Ax + By + C = 0$ where C is positive, then dividing by $\sqrt{A^2+B^2}$ and arranging the terms as in Art. IV-6(3), the equation can be written as

$$-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$$

which is in the perpendicular form.

Now proceeding as in (1), we obtain

$$d = -\frac{A}{\sqrt{A^2+B^2}}x_1 - \frac{B}{\sqrt{A^2+B^2}}y_1 - \frac{C}{\sqrt{A^2+B^2}}$$
$$= -\frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}.$$

Hence, the length of the perpendicular from (x_1, y_1) on the line $Ax + By + C = 0$ is

$$\mathbf{\frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}}$$

if we neglect the sign.

The length of the perpendicular from a point upon a line is therefore obtained by substituting the coordinates of the point in the left-hand member of the equation and dividing the expression by the square root of the sum of the squares of the coefficients of x and y.

IV-17 Sign of the perpendicular :

When we are concerned only with the length of the perpendicular we can ignore its sign. But there are cases where the sign is to be taken into account.

To fix the correct sign to the expression for the length of the perpendicular we adopt the convention that the perpendicular from the origin on a line is always positive.

If $Ax+By+C=0$ be the equation to the given line then the perpendicular from the origin O upon the line is positive. Also the perpendicular from a point $P(x_1, y_1)$ situated on the line itself, for which $Ax_1+By_1+C=0$, upon the line, is zero. Hence, we must expect that if the point crosses the line the perpendicular changes sign. We thus have perpendiculars from all points on the origin side of the line positive and those from the other side negative.

Consider the equation $Ax+By+C=0$. The perpendicular from the origin on this line is $\dfrac{A.0+B.0+C}{\sqrt{A^2+B^2}}=\dfrac{C}{\sqrt{A^2+B^2}}$.

Since, according to our convention this must be positive, the correct expression for the perpendicular is $+\dfrac{C}{\sqrt{A^2+B^2}}$ if C is positive and $-\dfrac{C}{\sqrt{A^2+B^2}}$ if C is negative.

Hence, if C is positive, the perpendicular from (x_1, y_1) $=+\dfrac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$ (in order that the perpendicular from the origin may be positive) if (x_1, y_1) is a point on the origin side of the line and $=-\dfrac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$ if (x_1, y_1) is a point on the other side.

IV-18. Lines bisecting the angles between two given lines :

Let the equations of the given lines AB and AC be

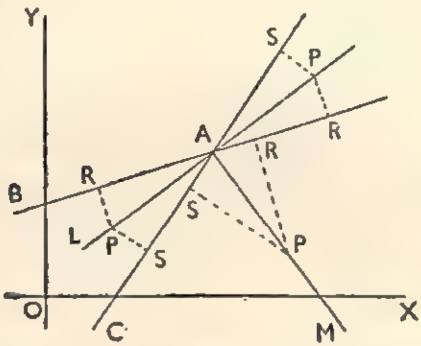
$$a_1x+b_1y+c_1=0$$

and $$a_2x+b_2y+c_2=0$$

where the equations are so written that c_1 and c_2 are both positive.

Let the bisectors of the angles between the lines be AL and AM and let $P(h, k)$ be any point on either of them.

If PR and PS be drawn perpendiculars on AB and AC respectively, then evidently these perpendiculars must be equal in magnitude.



Now, if P lies on AL the bisector of the angle in which the origin lies, the point P and the origin are on the same side of both AB and AC and hence, the perpendiculars are both positive, so that

$$\frac{a_1h+b_1k+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2h+b_2k+c_2}{\sqrt{a_2^2+b_2^2}} \qquad \cdots \quad (1)$$

(if P lies on LA produced, then P and O are on opposite sides of both AB and AC so that both the perpendiculars are negative and hence the relation (1) still holds).

Again, if P lies on AM the other bisector, the points P and O are on the same side of one of the lines and on opposite sides of the other and hence the perpendiculars are of opposite signs, so that

$$\frac{a_1h+b_1k+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2h+b_2k+c_2}{\sqrt{a_2^2+b_2^2}} \qquad \cdots \quad (2)$$

Hence, the required locus of P, *i.e.*, the pair of bisectors are given by the equations

$$\frac{\mathbf{a_1x+b_1y+c_1}}{\sqrt{\mathbf{a_1^2+b_1^2}}}=\pm\frac{\mathbf{a_2x+b_2y+c_2}}{\sqrt{\mathbf{a_2^2+b_2^2}}},$$

for, (*i*) being equations of the first degree in x and y they represent straight lines, and (*ii*) the point (h, k) lies on them, which is obvious from the relations (1) and (2).

Remark : When the equations are so written that c_1 and c_2 are both positive (or both negative) the positive sign corresponds to the bisector of the angle in which the origin lies.

WORKED OUT EXAMPLES

Ex. 1. *A line moves so that the algebraic sum of the perpendicular distances upon it from the three vertices of a triangle is always zero. Show that the line always passes through the centroid of the triangle.*

Let $x \cos \alpha + y \sin \alpha - p = 0$ ...(1) be the equation to the variable line and (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the vertices of the triangle.

From the given condition, we have

$$(x_1 \cos \alpha + y_1 \sin \alpha - p) + (x_2 \cos \alpha + y_2 \sin \alpha - p) + (x_3 \cos \alpha + y_3 \sin \alpha - p) = 0$$

i.e., $(x_1 + x_2 + x_3) \cos \alpha + (y_1 + y_2 + y_3) \sin \alpha - 3p = 0$,

i.e., $\dfrac{x_1 + x_2 + x_3}{3} \cos \alpha + \dfrac{y_1 + y_2 + y_3}{3} \sin \alpha - p = 0$,

i.e., $\bar{x} \cos \alpha + \bar{y} \sin \alpha - p = 0$ ... (2)

where $(\bar{x}, \bar{y})$ is the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3). [Worked out example 3, chap. I]

The relation (2) shows that the point $(\bar{x}, \bar{y})$ lies on the line (1), which proves the proposition.

Ex. 2. *Find the equations to the straight lines which bisect the angles between the lines*

$$2x + 3y + 4 = 0 \quad \text{and} \quad 6x - 4y - 5 = 0.$$

Writing the equations so that the numerical terms in both are positive the equations to the lines are

$$2x + 3y + 4 = 0$$

and $$-6x + 4y + 5 = 0.$$

The bisector of the angle in which the origin lies, is therefore given by

$$\frac{2x + 3y + 4}{\sqrt{2^2 + 3^2}} = \frac{-6x + 4y + 5}{\sqrt{6^2 + 4^2}},$$

i.e., $\quad \dfrac{2x+3y+4}{\sqrt{13}}=\dfrac{-6x+4y+5}{2\sqrt{13}},$

i.e., $\quad 2(2x+3y+4)=-6x+4y+5,$

i.e., $\quad 10x+2y+3=0.$

The equation to the other bisector is

$$\frac{2x+3y+4}{\sqrt{2^2+3^2}}=-\frac{-6x+4y+5}{\sqrt{6^2+4^2}}$$

i.e., $\quad 2(2x+3y+4)=6x-4y-5,$

i.e., $\quad 2x-10y-13=0.$

[Note that the pair of bisectors obtained are at right angles as they must necessarily be.]

EXERCISE IV(C)

1. In each of the following cases find whether the given point P is on the same or opposite sides of the given line L as the origin :

(*i*) $P(2,3)$, $\quad L\equiv 4x-5y+9=0$;

(*ii*) $P(-1,4)$ $\quad L\equiv 3x+y-2=0$;

(*iii*) $P(-3,-5)$, $\quad L\equiv x-2y-5=0$;

(*iv*) $P(0,7)$, $\quad L\equiv 5x-9y+47=0$;

(*v*) $P(2,-2)$, $\quad L\equiv 3x+4y+5=0$.

2. In each of the following cases find whether the given points P and Q lie on the same side or on opposite sides of the given line L :

(*i*) $P(2,3)$, $\quad Q(-5,-2)$, $\quad L\equiv 4x-5y+9=0$;

(*ii*) $P(-1,4)$, $\quad Q(2,-5)$, $\quad L\equiv 3x+y-2=0$;

(*iii*) $P(5,4)$ $\quad Q(4,-1)$ $\quad L\equiv 2x-3y+4=0$;

(*iv*) $P(1,1)$, $\quad Q(5,-3)$, $\quad L\equiv 3x+4y-5=0$;

(*v*) $P(0,-3)$, $\quad Q(-4,1)$, $\quad L\equiv 6x+7y+13=0$.

3. The vertices of a triangle are $(4, 5)$, $(-4, 3)$ and $(-1, -3)$. Prove, without plotting, that the origin is situated inside the triangle.

4. Find the length of the perpendicular

(*i*) from the origin on the line $3x+4y+5=0$;

(*ii*) from the point $(3, 1)$ on the line $5x-12y+1=0$;

(*iii*) from the point $(2,-3)$ on the line $15x-8y-3=0$;

(*iv*) from the point $(1,-1)$ on the line $2x\cos 45^\circ+y\sin 45^\circ+\sqrt{2}=0$;

(*v*) from the point $(3, 2)$ on the line $x\sin 30^\circ+y\cos 30^\circ=2$.

5. How far is the line $h(x+h)+k(y+k)=0$ from the origin ? [C. U.]

6. If p and p' be the perpendicular from the origin upon the lines

$$x \sin\theta + y \cos\theta = \frac{a}{2} \sin 2\theta,$$

and $$x \cos\theta - y \sin\theta = a \cos 2\theta,$$

prove that $4p^2 + p'^2 = a^2$. [C. U.]

7. What are the points on the axis of x whose perpendicular distance from the straight line

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ is } a\,?$$ [C. U. 1951]

8. Find the equations of the two straight lines drawn through the point $(0, a)$, on which the perpendiculars let fall from the point $(2a, 2a)$ are each of length a. [C. U. 1953]

9. Express the condition that the perpendicular dropped from the point $(3, -2)$ on the line $lx + my + n = 0$ may be of constant length 5. [C. U. 1956]

10. Find the distance between the two parallel lines :

(i) $15x - 8y + 3 = 0$ and $15x - 8y + 20 = 0$;

(ii) $3x + 4y - 15 = 0$ and $6x + 8y + 15 = 0$.

11. Find the equations to the straight lines bisecting the angles between the following pairs of straight lines, stating which of them bisects the angle in which the origin lies :

(i) $3x - 4y + 7 = 0$, $7x + 24y + 5 = 0$;

(ii) $2y = 3x - 1$, $3y = 2x + 1$;

(iii) $4x + 3y + 2 = 0$, $5x - 12y - 1 = 0$.

12. Find the equations of the straight lines passing through the foot of the perpendicular from the point (h, k) upon the straight line $Ax + By + C = 0$ and bisecting the angle between the perpendicular and the straight line. [C. U.]

13. In any triangle, prove that

(i) the three internal bisectors are concurrent ;

(ii) the external bisectors of two angles and the internal bisector of the third angle are concurrent.

[**Hints :** Take the origin inside the triangle and the equations to the sides $x \cos \alpha_r + y \sin \alpha_r = p_r$ $(r = 1, 2, 3)$.]

14. Find the internal bisectors of the angles of the triangle whose sides are $x = 0$, $y = 0$ and $3x + 4y - 12 = 0$. [C. U.]

15. Show that the perpendiculars let fall from any point of the straight line $7x - 9y + 10 = 0$ upon the two straight lines $3x + 4y = 5$ and $12x + 5y = 7$ are equal to each other. [C. U. 1952]

Answers :

1. (*i*) same ; (*ii*) same ; (*iii*) opposite ; (*iv*) opposite ; (*v*) same.
2. (*i*) opposite ; (*ii*) same ; (*iii*) same ; (*iv*) opposite ; (*v*) same.
4. (*i*) 1 ; (*ii*) $\frac{4}{13}$; (*i* 3 ; (*iv*) $\frac{3}{\sqrt{5}}$; (*v*) $\sqrt{3}-\frac{1}{2}$.
5. $\sqrt{h^2+k^2}$. 7. $\left\{\frac{a}{b}(b\pm\sqrt{a^2+b^2}), 0\right\}$. 8. $y=a$, $4x-3y+3a=0$.
9. $(3l-2m+n)^2=25(l^2+m^2)$. 10. (*i*) 1 ; (*ii*) $4\frac{1}{3}$.
11. (*i*) $4x-22y+15=0$ and $11x+2y+20=0$, $4x-22y+15=0$;

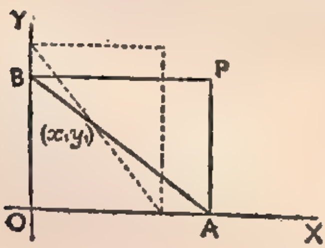
 (*ii*) $x-y=0$ and $x+y-2=0$, $x-y=0$;

 (*iii*) $11x-3y+3=0$ and $27x+99y+31=0$, $11x-3y+3=0$.
12. $B(x-h)-A(y-k)=\pm(Ax+By+C)$.
14. $x=y$, $x+3y-4=0$, $2x+y-3=0$.

IV-19. Locus Problems :

Problem 1. *A variable line passes through a fixed point (x_1, y_1) and meets the axes in A and B. If the rectangle OAPB is completed, find the locus of P.*

Let the coordinates of P any point on the locus be (h, k).



Then $OA=h$ and $OB=AP=k$
The equation to the line AB is

then $\frac{x}{h}+\frac{y}{k}=1$.

For all positions of P, this line must pass through the fixed point (x_1, y_1).

Hence, $\frac{x_1}{h}+\frac{y_1}{k}=1$.

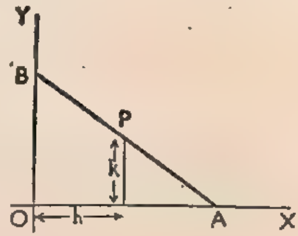
$\therefore$ (h, k) satisfies the equation

$$\frac{x_1}{x}+\frac{y_1}{y}=1$$

which is therefore the required equation to the locus.

Problem 2. *A line moves so that the sum of the intercepts made by it on the axes is always constant. Find the locus of the middle point of the portion of the line intercepted between the axes.*

Let AB be one position of the variable line and $P(h, k)$ its middle point. Clearly $OA=2h$ and $OB=2k$. From the given condition, we have



$$2h+2k=\text{constant}=2l \text{ (say)}$$

i.e., $h+k=l$

$\therefore$ (h, k) lies on the locus given by

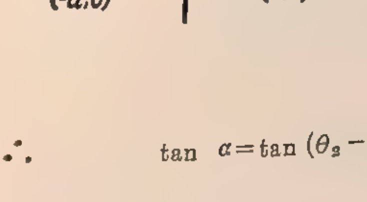
$$x+y=l$$

which is a straight line.

Problem 3. *The base of a triangle is fixed. Find the locus of the vertex, when it moves so that the vertical angle is always constant.*

Let AB be the fixed base of length $2a$ and let α be the constant vertical angle.

We choose O, the middle point of AB as origin, OB the axis of x and a line through O perpendicular to AB as the axis of y.



We have clearly $OA=OB=a$, so that the points A and B are respectively $(-a, 0)$ and $(a, 0)$

Let $P(x, y)$ be any point on the locus. If then θ_1 and θ_2 are the angles which AP and BP make with the x-axis, we have

$$\alpha=\theta_2-\theta_1$$

$$\therefore \qquad \tan\alpha=\tan(\theta_2-\theta_1)=\frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_2\tan\theta_1}$$

where $\qquad \tan\theta_1=\text{gradient of } AP=\dfrac{y}{x+a}$

and $\qquad \tan\theta_2=\text{gradient of } BP=\dfrac{y}{x-a}.$

Hence $$\tan\alpha=\frac{\dfrac{y}{x-a}-\dfrac{y}{x+a}}{1+\dfrac{y}{x-a}\cdot\dfrac{y}{x+a}}=\frac{2ay}{x^2+y^2-a^2},$$

i.e., $$x^2+y^2-a^2=2ay\cot\alpha,$$

or, $$x^2+y^2-2ay\cot\alpha-a^2=0$$

which being the relation between the coordinates of any point on the locus is the required equation to the locus.

EXERCISE IV (D)

1. From any point P on the line AB which cuts off equal intercepts $2a$ from the axes perpendiculars PR and PS are drawn on the axes. Find the locus of the middle point of RS.

2. A variable line passes through a fixed point (α, β) and meets the axes in A and B. Find the locus of the mid-point of AB.

3. A straight line AB of constant length l slides between two rectangular axes OX and OY. If the rectangle $OAPB$ is completed, find the locus of P.

4. A variable line passes through the fixed point (α, β). Find the locus of the foot of the perpendicular from the origin on the line.

5. A variable point P is joined to two fixed points $A(-a, 0)$ and $B(a, 0)$. If the sum of the gradients of AP and BP is unity, find the locus of P.

6. The extremities of the base of a triangle PAB are $A\ (-a, 0)$ and $B(a, 0)$. Find the locus of the vertex P when $\angle PBA=2\angle PAB$.

7. A line passing through a fixed point $A(-a, 0)$ meets another line through a second fixed point $B(a, 0)$ at P. Find the locus of P, when
   (*i*) AP and BP are at right angles ;
   (*ii*) the product of the gradients of AP and BP is unity.

8. A point P moves so that the perpendiculars from it to the lines $x+2y=1$ and $4x-2y=3$ are equal. Find the locus of P.

9. Perpendiculars are drawn from a point P on the sides of a triangle. If the perpendiculars be p, q, r and if these be connected by the relation $lp+mq+nr=0$, where l, m, n are constants, then prove that the locus of P is a straight line.

10. If the sum of the perpendiculars dropped from a variable point P on the two lines $x+y-5=0$ and $3x-2y+7=0$ be equal to 10, prove that P must move on a right line. [*C. U.* 1950]

Answers :

1. $x+y=a$. 2. $\dfrac{\alpha}{2x}+\dfrac{\beta}{2y}=1$. 3. $x^2+y^2=l^2$.
4. $x^2+y^2-\alpha x-\beta y=0$. 5. $x^2-2xy-a^2=0$. 6. $y^2-3x^2-2ax+a^2=0$.
7. (*i*) $x^2+y^2=a^2$; (*ii*) $x^2-y^2=a^2$. 8. $2x-6y=1$, $6x+2y=5$.

CHAPTER V

*TANGENTS AND NORMALS

V-1. The symbols δx, δy and $\dfrac{dy}{dx}$:

We shall find it convenient to use the symbols δx and δy—read as 'delta x' and 'delta y'—to mean small increments in the variables x and y and also to denote by the symbol $\dfrac{dy}{dx}$ the limiting value of the ratio $\dfrac{\delta y}{\delta x}$ as δx tends to zero, *i.e.*, the value to which the ratio continually approaches as its limit as δx is taken closer and closer to zero.

This is written as

$$\underset{\delta x \to 0}{Lt}\ \frac{\delta y}{\delta x}=\frac{dy}{dx}.$$

V-2. Gradient of a Curve :

It has been shown in Art. IV-2 that the gradient of a straight line is the ratio of the increase in the ordinate of a point to the corresponding increase in the abscissa as the point passes from one position to another on the line, and that it is independent of the size of the increments and also of the position of the point.

We now proceed to define the gradient of a curve at a point of it and see how it differs from that of a straight line.

Consider the curve given by the equation

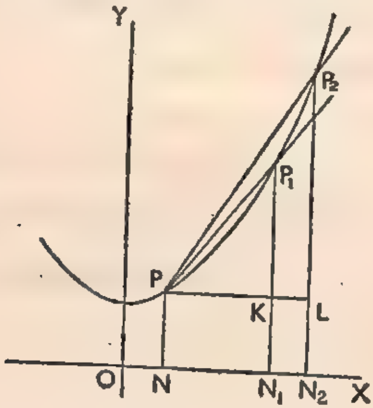
$$y=ax^2+b$$

Let P be a point on the curve and P_1, P_2 two other points on it. Draw the ordinates PN, P_1N_1 and P_2N_2 and also draw PKL parallel to OX meeting the ordinates P_1N_1 and P_2N_2 in K and L respectively.

Now the gradient of the chord PP_1 is $\dfrac{KP_1}{PK}$ and that of the chord PP_2 is $\dfrac{LP_2}{PL}$. These two ratios are clearly not equal, so

* This chapter may be omitted at the first reading.

that the gradient of the chord passing through P changes with the position of the other extremity of the chord.



If (x, y) be the coordinates of the point P and $(x+\delta x, y+\delta y)$ be the coordinates of P_1, so that the increment δx in x is NN_1, *i.e.*, PK and the corresponding increment δy in y is KP_1, then the gradient of the chord $PP_1 = \dfrac{KP_1}{PK} = \dfrac{\delta y}{\delta x}$.

Now, since P_1 is a point on the curve $y = ax^2 + b$, we have

$$y + \delta y = a(x + \delta x)^2 + b$$
$$= ax^2 + 2ax\delta x + a\delta x^2 + b.$$

But $$y = ax^2 + b.$$

$$\therefore \quad \delta y = 2ax\delta x + a\delta x^2.$$

Hence, $$\frac{\delta y}{\delta x} = 2ax + a\delta x$$

i.e., gradient of the chord $PP_1 = 2ax + a\delta x$ which thus depends on the position of the point on the curve and the magnitude of the increment in x. As P_1 approaches P, δx approaches zero and the gradient tends to the value $2ax$. We say the gradient of the curve at any point $P\,(x, y)$ on it is $2ax$.

At a given point, say $(x_1\ y_1$ the gradient of the curve is therefore $2ax_1$.

Def : The gradient of a curve varies from point to point and at any point (x, y) it is defined as the limiting value of the ratio $\dfrac{\delta y}{\delta x}$ as δx tends to zero, which is denoted by the symbol $\dfrac{dy}{dx}$.

V-3. Illustrative Examples :

A few examples to illustrate the method of finding the gradient of a curve at a given point are given below :

Example 1. *Find the gradient of the curve whose equation is $y=x^3$ at the point whose abscissa is a.*

If P be the point (x, y) and P_1 be $(x+\delta x,\ y+\delta y)$, then since P_1 is a point on the curve, we have

$$y+\delta y=(x+\delta x)^3$$
$$=x^3+3x^2\delta x+3x\delta x^2+\delta x^3.$$

But $\qquad y=x^3.$

$\therefore\quad \delta y=3x^2\delta x+3x\delta x^2+\delta x^3.$

Hence, $\qquad \dfrac{\delta y}{\delta x}=3x^2+3x\delta x+\delta x^2.$

$$\therefore\quad \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=3x^2.$$

Hence, at the point whose abscissa is a

i.e., $\quad$ when $x=a$, $\qquad$ the gradient$=3a^2$.

Example 2. *Find the gradient of curve $x^2+y^2=a^2$ at the point (x_1, y_1).*

If on the curve, P be the point (x, y) and P_1 be $(x+\delta x, y+\delta y)$, then we have

$$(x+\delta x)^2+(y+\delta y)^2=a^2,$$

i.e., $\quad x^2+2x\delta x+\delta x^2+y^2+2y\delta y+\delta y^2=a^2.$

But $\quad x^2+y^2=a^2.$

$\therefore\quad \delta x(2x+\delta x)+\delta y(2y+\delta y)=0,$

i.e., $\quad \dfrac{\delta y}{\delta x}=-\dfrac{2x+\delta x}{2y+\delta y}.$

$$\therefore\quad \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\frac{2x}{2y}=-\frac{x}{y},$$

since, when δx approaches zero δy also approaches zero.

Hence, the gradient at (x_1, y_1) is $-\dfrac{x_1}{y_1}$.

Example 3. *Prove that the gradient of the curve $xy=c^2$ at the point (x_1, y_1) is $-\dfrac{y_1}{x_1}$*

The equation to the curve may be written as

$$y=\frac{c^2}{x}.$$

If therefore (x, y) and $(x+\delta x, y+\delta y)$ be two points on the curve near to one another, we have

$$y+\delta y=\frac{c^2}{x+\delta x}.$$

$$\therefore \quad \delta y=(y+\delta y)-y$$

$$=\frac{c^2}{x+\delta x}-\frac{c^2}{x}$$

$$=\frac{-c^2\delta x}{x(x+\delta x)}.$$

Hence, $\dfrac{\delta y}{\delta x}=-\dfrac{c^2}{x(x+\delta x)}.$

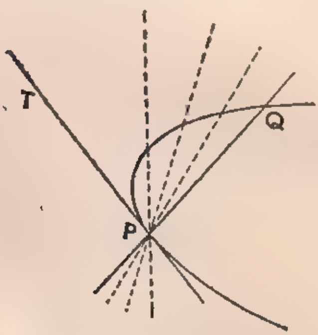
$$\therefore \quad \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\frac{c^2}{x^2}=-\frac{xy}{x^2}\text{(from the equation)}=-\frac{y}{x}.$$

Hence, at the point (x_1, y_1) the gradient $=-\dfrac{y_1}{x_1}$.

Note. Students acquainted with the differential notation will be able to find out at once $\dfrac{dy}{dx}$ for any given curve and hence obtain the gradient at a point of it.

V-4. Tangent to a curve :

Let P be a point on a curve and Q be another point near P.



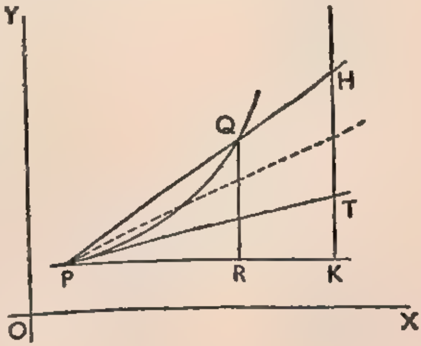
The line PQ is a secant. If the point Q be now made to move along the curve until it takes up a position next to P, in other words, when Q tends to coincidence with P, in that case PQ tends to the limiting position PT and we say PT is the tangent to the curve at P. We thus arrive at the following definition of a tangent.

Def. : *The tangent to a curve at a point on it is the limiting position of the secant passing through the point when the other point of intersection tends to coincidence with the first.*

V-5. Gradient of the tangent :

We shall now proceed to prove the relation that exists between the gradient of a curve at a point and the gradient of the tangent to the curve at that point.

Let PQ be a chord of a curve and PT the tangent to it at P.



Through P and Q draw PR and QR parallels respectively to OX and OY and through a point T on the tangent PT draw HTK parallel to OY meeting PQ in H and PR in K.

Now the gradient of the curve at P

$= \text{Limiting value of } \dfrac{PQ}{PR}$

as Q approaches P

$= \text{Limiting value of } \dfrac{KH}{PK}$ as Q approaches $P = \dfrac{KT}{PK}$

for, as Q approaches P, the chord PQ approaches the tangent PT so that H approaches T and hence, KH tends to the value KT.

But $\dfrac{KT}{PK}$ gives the gradient of the tangent PT.

Hence, we conclude that

The gradient of a curve at any point of it is the same as the gradient of the tangent to the curve at that point.

V-6. Equation of the tangent :

The problem of finding the equation of the tangent to a curve at a given point of it now resolves itself into forming the equation to a straight line passing through the given point and having a gradient equal to the gradient of the curve at that point. If (x_1, y_1) be the given point on the curve then the equation to the tangent at (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

where m is the gradient of the curve at (x_1, y_1).

For example :

(1) The equation of the tangent to the curve $y=x^3$ at the point whose abscissa is a is

$$y-a^3=3a^2(x-a)$$

for the ordinate of the point whose abscissa is a is a^3 from the equation to the curve, and the gradient of the curve at this point is $3a^2$ (*Art. V-3. Ex.* 1).

Hence, the required equation of the tangent is

$$y=3a^2x-2a^3.$$

(2) The equation of the tangent to the curve

$$2x^2+3y^2=5$$

and the point (1, 1) is

$$y-1=m(x-1)$$

where $$m=\frac{dy}{dx} \text{ at } (1, 1).$$

Now, $2(x+\delta x)^2+3(y+\delta y)^2=5.$

Also $2x^2+3y^2=5.$

$\therefore$ $4x\delta x+2\delta x^2+6y\delta y+3\delta y^2=0,$

or, $2\delta x(2x+\delta x)+3\delta y(2y+\delta y)=0,$

or, $\dfrac{\delta y}{\delta x}=-\dfrac{2(2x+\delta x)}{3(2y+\delta y)}.$

Hence, $\dfrac{dy}{dx}=\lim\limits_{\delta x\to 0}\dfrac{\delta y}{\delta x}=-\dfrac{4x}{6y}$ [since as δx approaches zero, δy also approaches zero]

$$=-\frac{2x}{3y}.$$

Hence, $\dfrac{dy}{dx}$ at $(1, 1)=-\frac{2}{3}$.

The required equation of the tangent is therefore

$$y-1=-\tfrac{2}{3}(x-1),$$

or, $$2x+3y=5.$$

V-7. Normal :

Def : The normal to *a* curve at a point of it is a straight line through the point perpendicular to the tangent at the point.

Equation of the normal :

The equation of the normal can at once be obtained by finding the gradient of the tangent and applying the condition of perpendicularity.

If m' be the gradient of the normal and m that of the tangent, we have

$$mm' = -1. \quad \therefore \quad m' = -\frac{1}{m}.$$

Hence, the equation of the normal at (x_1, y_1) to a curve is given by

$$y - y_1 = -\frac{1}{m}(x - x_1).$$

For example, to find the equation of the normal to the curve $xy = c^2$ at the point (x_1, y_1), we have (*from Ex. 3, Art. V-3*) the gradient of the tangent at (x_1, y_1) is $-\dfrac{y_1}{x_1}$ and therefore the equation of the normal at (x_1, y_1) is $y - y_1 = \dfrac{x_1}{y_1}(x - x_1)$, or, $yy_1 - xx_1 = y_1^2 - x_1^2$.

EXERCISE V

1. Find the gradient at any point $P(x, y)$ of the following curves :

 (*i*) $y = (x+1)^2$; (*ii*) $y = \dfrac{1}{x^2}$; (*iii*) $y^2 = 4x$;

 (*iv*) $x^2 - y^2 = a^2$; (*v*) $y = 4x^3 - 3x^2 + x$.

2. Find the gradient and the equation of the tangent to the following curves at the point P :

 (*i*) $y^2 = x, P(1,1)$; (*ii*) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 2, P(3, -2)$;

 (*iii*) $xy = 1, P(\frac{2}{3}, \frac{3}{2})$; (*iv*) $x^2 - 2y^2 = 2, P(2, 1)$;

 (*v*) $x^2 + y^2 = 13, P(2, 3)$.

3. Find the equations of the tangent and normal at P for the following curves :

 (*i*) $x^2 + y^2 = a^2, P(a\cos\theta, a\sin\theta)$;

 (*ii*) $y^2 = 4ax, P(at^2, 2at)$;

 (*iii*) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, P(a\cos\phi, b\sin\phi)$;

 (*iv*) $x^2 - y^2 = a^2, P(a\sec\phi, a\tan\phi)$;

 (*v*) $xy = c^2, P\left(ct, \dfrac{c}{t}\right)$.

4. Find the equation of the tangent to the curve $xy = \frac{1}{2}a^2$ at a point $P(x_1, y_1)$ on it, and prove that for all positions of P the area of the triangle cut off from the coordinate axes by the tangent is constant.

5. Find the coordinates of the points on the curve

$$y = 2x^3 - 3x^2 - 12x + 10$$

at which the tangent is parallel to the axis of x.

6. Find the equations of the tangent and normal to the curve $2y^2 = 9x$ at the point $P(2,3)$. If the tangent and normal meet the axis of x at T and G respectively, then prove that

$$ST = SG = SP$$

where S is a point on the axis of x at a distance $\frac{9}{8}$ from the origin.

Answers :

1. (i) $2x+2$; (ii) $-\dfrac{2}{x^3}$; (iii) $\dfrac{2}{y}$; (iv) $\dfrac{x}{y}$; (v) $12x^2 - 6x + 1$.

2. (i) $\frac{1}{2}$, $x - 2y + 1 = 0$; (ii) $\frac{2}{3}$, $2x - 3y = 12$; (iii) $-\frac{9}{4}$, $9x + 4y = 12$;
   (iv) 1, $x - y = 1$; (v) $-\frac{2}{3}$, $2x + 3y = 13$.

3. (i) $x \cos\theta + y \sin\theta = a$, $y = x \tan\theta$;
   (ii) $yt = x + at^2$, $y + tx = 2at + at^3$;
   (iii) $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1$, $ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$;
   (iv) $x \sec\phi - y \tan\phi = a$, $x \cos\phi + y \cot\phi = 2a$;
   (v) $\dfrac{x}{t} + yt = 2c$, $t^3x - ty = ct^4 - c$.

4. $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$, area $= a^2$. 5. $(2, -10)$, $(-1, 17)$.

6. $3x - 4y + 6 = 0$, $4x + 3y - 17 = 0$.

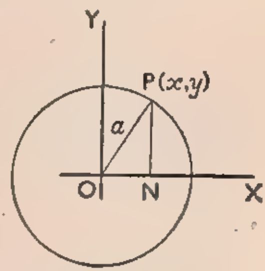
CHAPTER VI

THE CIRCLE

VI-1. Formation of equations :

(A) *To find the equation of a circle whose centre is at the origin and whose radius is a.*

Let $P(x, y)$ be any point on the circumference of the circle. Draw the ordinate PN and join OP.



Then the geometrical condition satisfied by P in order that it may be a point on the locus is

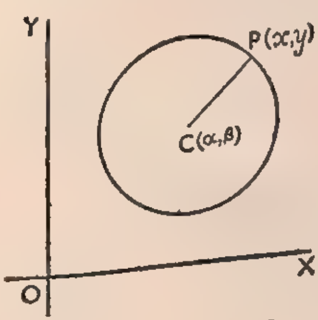
$$OP = a$$

which gives $ON^2 + NP^2 = a^2$

i.e., $\mathbf{x^2 + y^2 = a^2}$.

This being the relation between the coordinates of any point on the circle is the required equation of the circle.

(B) *To find the equation of a circle whose centre is at a given point (α, β) and whose radius is a.*



Let $P(x, y)$ be any point on the circle whose centre is $C(\alpha, \beta)$.

We then have

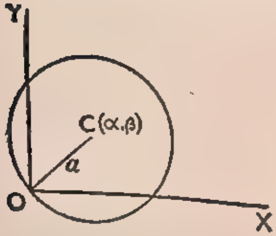
$$CP^2 = y^2$$

$$\therefore\ \mathbf{(x-\alpha)^2 + (y-\beta)^2 = a^2}$$

which is the required equation of the circle.

Cor. If the centre is at the origin, $\alpha = 0$, $\beta = 0$, and the equation reduces to $x^2 + y^2 = a^2$.

VI-2. Some particular cases :



(1) If the origin is on the circumference, we have $OC=a$.

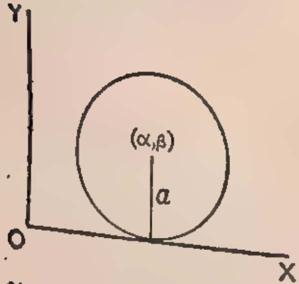
$$\therefore \quad \alpha^2+\beta^2=a^2.$$

Hence, the equation

$$(x-\alpha)^2+(y-\beta)^2=a^2$$

reduces to

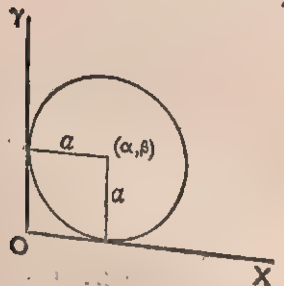
$$x^2+y^2-2\alpha x-2\beta y=0.$$



(2) If the circle touches the axis of x, we have $\beta=a$ and the equation to the circle becomes

$$(x-\alpha)^2+(y-\beta)^2=\beta^2,$$

$$i.e.,\ x^2+y^2-2\alpha x-2\beta y+\alpha^2=0.$$



(3) If the circle touches both the axes, we have

$$\alpha=\beta=a$$

and the equation to the circle becomes

$$(x-a)^2+(y-a)^2=a^2,$$

$$i.e.,\ x^2+y^2-2ax-2ay+a^2=0.$$

VI-3. The general equation :

The equation

$$(x-\alpha)^2+(y-\beta)^2=a^2 \qquad \ldots \quad (1)$$

can, by properly choosing α, β and a be made to represent any circle. Hence, equation (1) can be looked upon as the **general equation** of a circle. The equation when simplified is

$$x^2+y^2-2\alpha x-2\beta y+\alpha^2+\beta_2-a^2=0.$$

Replacing $-\alpha, -\beta$ and $\alpha^2+\beta^2-a^2$ by the constants g, f and c respectively, it can be written as

$$x^2+y^2+2gx+2fy+c=0$$

which may be taken as another form of the general equation.

Hence, the general equation of a circle may be taken in either of the following forms :

(*i*) $\mathbf{(x-\alpha)^2+(y-\beta)^2=a^2}$;

(*ii*) $\mathbf{x^2+y^2+2gx+2fy+c=0}$.

VI-4. The centre and radius of the circle given by the general equation :

The equation

$$x^2+y^2+2gx+2fy+c=0 \quad \dots \quad (1)$$

can be written as

$$(x+g)^2+(y+f)^2=g^2+f^2-c,$$

i.e., $\{x-(-g)\}^2+\{y-(-f)\}^2=(\sqrt{g^2+f^2-c})^2$

from which it is clear that

(*i*) **the centre of the circle is the point $(-g, -f)$;**

(*ii*) **the radius of the circle is $\sqrt{g^2+f^2-c}$.**

Hence in order that the equation (1) may represent a real circle, g^2+f^2-c must be greater than zero.

Note : If $g^2+f^2-c < 0$, the radius is imaginary and hence it is an imaginary circle. The equation in this case has no geometrical significance but analytically it represents a circle with a real centre and an imaginary radius.

If, $g^2+f^2-c=0$, the radius is zero. In this case the circle reduces to a point coinciding with the centre $(-g, -f)$. Such a circle is known as a **point-circle.** In fact, the equation in this case reduces to $(x+g)^2+(y+f)^2=0$, which gives $x+g=0$ and $y+f=0$, since a square quantity cannot be negative. These equations determine the only point $(-g, -f)$.

VI-5. Condition that the general equation of the second degree may represent a circle :

The general equation of the second degree in x and y is

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

which involves all possible terms in x and y in a degree not higher than 2.

If in this equation $a=b$ and $h=0$, the equation reduces to

$$ax^2+ay^2+2gx+2fy+c=0$$

which on division by a becomes

$$x^2+y^2+2\frac{g}{a}x+2\frac{f}{a}y+\frac{c}{a}=0,$$

i.e., $$x^2+y^2+2g'x+2f'y+c'=0$$

where g', f' and c' are constants.

The above equation being identical in form with the general equation of a circle, we conclude that

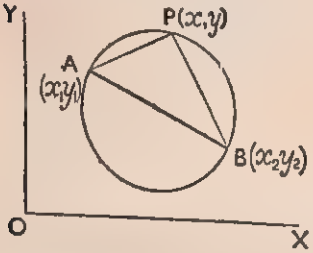
The general equation of the second degree in x and y, viz.,

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a circle if $\mathbf{a}=\mathbf{b}$ *and* $\mathbf{h}=\mathbf{0}$, *i.e., the coefficients of* x^2 *and* y^2 *are the same and the term containing xy is absent.*

VI-6. Circle having two given points as extremities of a diameter :

To find the equation of the circle described on the line joining two given points as diameter.



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points and let $P(x, y)$ be any point on the circle.

Now the geometrical condition satisfied by P in order that it may be a point on the required locus is that $\angle APB=$a right angle.

Now the gradient of the line $AP=\dfrac{y-y_1}{x-x_1}$

the gradient of the line $BP=\dfrac{y-y_2}{x-x_2}$.

$\therefore$ From the above condition,

$$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1,$$

i.e., $\quad(\mathbf{x}-\mathbf{x}_1)(\mathbf{x}-\mathbf{x}_2)+(\mathbf{y}-\mathbf{y}_1)(\mathbf{y}-\mathbf{y}_2)=\mathbf{0}.$

This is therefore the required equation of the circle.

VI-7. Circle through three given points :

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the three given points and let

$$x^2+y^2+2gx+2fy+c=0 \qquad \cdots \qquad (1)$$

be the circle which passes through them.

Then clearly,

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \cdots \qquad (2)$$

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad \cdots \qquad (3)$$

and

$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \qquad \cdots \qquad (4)$$

From equations (2), (3) and (4) the constants g, f and c can be determined and on substitution of these values in equation (1) we get the required equation of the circle.

N.B. The method should be remembered to solve problems in particular cases.

VI-8. Constants of the equation to a circle :

The general equation of the circle, *viz.*,

$$x^2+y^2+2gx+2fy+c=0$$

contains three arbitrary constants—g, f and c. If these constants be found out, the radius and the centre of the circle will be known and the circle represented by the equation will be definitely known. Now to find out the three constants, we require three equations connecting them, and to obtain the three equations, three independent geometrical conditions satisfied by the circle must be given. For example, if three points on a circle be given the equation of the circle can be found out as in Art. VI-7

WORKED OUT EXAMPLES

Ex. 1. *Find the centre and radius of the circle*

$$9x^2+9y^2+36x-24y-29=0.$$

Dividing both sides by 9,

we have $$x^2+y^2+4x-\tfrac{8}{3}y-\tfrac{29}{9}=0$$

which can be written as

$$(x+2)^2+(y-\tfrac{4}{3})^2=2^2+(\tfrac{4}{3})^2+\tfrac{29}{9},$$

i.e., $$(x+2)^2+(y-\tfrac{4}{3})^2=3^2.$$

Hence, the centre of the circle is the point $(-2, \tfrac{4}{3})$ and the radius is 3.

Ex. 2. *Find the equation to the circle whose centre is the point* $(-4, 5)$ *and which passes through the point* (1, 2.)

Since (1, 2) is a point on the circumference of the circle, its radius

$$=\sqrt{(1+4)^2+(2-5)^2}=\sqrt{25+9}=\sqrt{34}.$$

Hence, the required equation is

$$(x+4)^2+(y-5)^2=34,$$

i.e., $$x^2+y^2+8x-10y+7=0$$

Otherwise: Let r be the radius of the circle. Then its equation is

$$(x+4)^2+(y-5)^2=r^2.$$

Since, the circle passes through the point (1,2), we must have

$$(1+4)^2+(2-5)^2=r^2,$$

i.e., $$r^2=34.$$

Hence, the required equation is

$$(x+4)^2+(y-5)^2=34,$$

i.e., $$x^2+y^2+8x-10y+7=0.$$

Ex. 3. *Find the equation to the circle which passes through the points* (1, 2), (−3, 4), *and* (5, −6).

Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0. \qquad \cdots \quad (1)$$

Since it passes through (1, 2), (−3, 4) and (5, −6), we have

$$2g+4f+c+5=0$$

$$-6g+8f+c+25=0$$

and $$10g-12f+c+61=0.$$

Solving these equations, we get $g=\frac{17}{8}$, $f=\frac{19}{8}$ and $c=-\frac{125}{8}$.

Substituting these values in (1) the required equation is

$$3x^2+3y^2+34x+38y-125=0.$$

Ex. 4. *Find the equation to the circle which is concentric with the circle* $x^2+y^2-4x+6y-3=0$ *and which passes through the point* (5, −2).

The equation to any concentric circle is clearly of the form

$$x^2+y^2-4x+6y+k=0.$$

It passes through the point (5, −2) if

$$5^2+(-2)^2-4.5+6.(-2)+k=0.$$

i.e., if $$25+4-20-12+k=0,$$

i.e., if $$k=3.$$

Hence, the required equation to the circle is

$$x^2+y^2-4x+6y+3=0.$$

EXERCISE VI(A)

1. Find the equation to the circle whose centre is C and radius r, when

(i) C is $(0, 0)$, $r=4$;

(ii) C is $(2, ?)$, $r=5$;

(iii) C is $(-3, 4)$, $r=1$;

(iv) C is $(-5, -7)$, $r=9$;

(v) C is $(0, a)$, $r=a$.

2. Find the coordinate of the centre and the radius of the following circles ;

(i) $x^2+(y+3)^2=16$;
(ii) $x^2+y^2-2x+4y-4=0$;
(iii) $12x^2+12y^2-12x+8y+3=0$;
(iv) $x^2+y^2+4x-6y+13=0$;
(v) $5x^2+5y^2-4x-3y=15$.

3. Find the equation to the circle circumscribing the triangle whose vertices are

(i) $(0, 0)$, $(3, 0)$, $(0, 5)$;
(ii) $(1, 7)$, $(-2, 6)$, $(5, 5)$;
(iii) $(7, -3)$, $(5, 11)$, $(-9, -3)$.

4. (i) Find the equation of the circle which passes through $(2, -1)$, $(2, 3)$ and $(4, -1)$ and obtain its radius and the coordinates of the centre. [*C. U.*]

(ii) Find the equation to the circle which goes through the three points $(5, 3)$, $(6, -4)$ and $(-1, -5)$. Also find the centre of this circle. [*C. U.* 1957]

5. Prove that the following sets of four points are concyclic, and find the equations to the circles passing through them ;

(i) $(1, 1)$, $(2, 0)$, $(3, -3)$, $(-5, -7)$;

(ii) $(8, 4)$, $(6, 8)$, $(-1, 7)$, $(-2, 4)$.

6. Obtain the equation of the circle whose centre is the point $(2, 3)$ and which passes through the point $(5, 7)$. [*C. U.*]

7. Find the equation to the circle whose centre is at the origin and which meets the straight line $\frac{x}{5}-\frac{y}{6}=1$, on the axis of y. [*C. U.*]

8. Find the equation to the circle whose centre is the point $(2, -3)$ and which passes through the intersection of the straight lines

$$2x-3y=23 \text{ and } 3x+5y+13=0.$$

9. Prove that the centres of the three circles $x^2+y^2=1$, $x^2+y^2+6x-2y=1$, $x^2+y^2-12x+4y=1$ lie on a right line. [*C. U.*]

10. Determine the centres and radii of the three circles $x^2+y^2-2x+2y-7=0$, $x^2+y^2-6x-2y-6=0$ and $x^2+y^2-8x-4y-5=0$ and verify that the centres lie on a right line whose equation you are required to obtain. [*C. U.* 1950]

11. Find the equation of the circle which touches the coordinate axes at (1, 0) and (0, 1). [C. U.]

12. Obtain the equation of the circle which passes through the two points on the axis of x which are at a distance 2 from the origin and whose radius is 5. [C. U. 1952]

13. Find the equation to the circle which touches both axes and whose radius is 3.

14. Find the equation to the circle which passes through the origin and cuts off intercepts 5 and 7 from the positive directions of the axes of x and y.

15. Find the equation to the circle which touches both axes and passes through the point (1, 3).

16. Find the equation to the circle which touches axis of y and passes through the two points (2, 3) and (4, 1).

17. Find the equation to the circle which passes through the points (3, 4), (−1, 6) and has its centre on the straight line $3x+5y-2=0$.

18. $ABCD$ is a square whose side is a; taking AB and AD as axes prove that the equation to the circle circumscribing the square is $x^2+y^2=a(x+y)$. [C. U. 1951]

19. Find the equation to the circle described on the join of (7, −5) and (−3, 9) as diameter.

20. Find the equation to the circle which is concentric with the circle whose equation is

$$2x^2+2y^2-7x+9y+10=0$$

and which passes through the point (1, 1).

21. Prove that the two circles

(i) $x^2+y^2+4x-10y-20=0,\quad x^2+y^2-8x+6y+16=0$;

(ii) $x^2+y^2+4x-6y-36=0,\quad x^2+y^2+10x+2y+22=0$

touch each other, externally in (i) and internally in (ii) and in each case determine the coordinates of the point of contact.

[Hints: The point of contact divides the line joining the centres in the ratio of the radii].

22. Find the equation of the circle which touches the x-axis, passes through the point (1, 1) and whose centre lies in the first quadrant on the line $x+y=3$. [C. U.]

23. Find the area of the equilateral triangle inscribed in the circle

$$x^2+y^2+2gx+2fy+c=0.$$

[C. U.]

Answers :

1. (*i*) $x^2+y^2=16$; (*ii*) $x^2+y^2-4x-6y-12=0$;
(*iii*) $x^2+y^2+6x-8y+24=0$; (*iv*) $x^2+y^2+10x+14y-7=0$;
(*v*) $x^2+y^2-2ay=0$.

2. (*i*) $(0, -3), 4$; (*ii*) $(1, -2), 3$; (*iii*) $(\frac{1}{2}, -\frac{1}{3}), \frac{1}{3}$;
(*iv*) $(-2, 3), 0$; (*v*) $(\frac{2}{5}, \frac{3}{10}), \frac{\sqrt{13}}{2}$.

3. (*i*) $x^2+y^2-3x-5y=0$; (*ii*) $x^2+y^2-2x-4y-20=0$;
(*iii*) $x^2+y^2+2x-6y-90=0$.

4. (*i*) $x^2+y^2-6x-2y+5=0$, $\sqrt{5}$, $(3, 1)$;
(*ii*) $x^2+y^2-4x+2y-20=0$, $(2, -1)$.

5. (*i*) $x^2+y^2+4x+6y-12=0$; (*ii*) $x^2+y^2-6x-8y=0$.

6. $x^2+y^2-4x-6y-12=0$. 7. $x^2+y^2=36$.

8. $x^2+y^2-4x+6y+5=0$. 10. $x-y=2$.

11. $x^2+y^2-2x-2y-1=0$. 12. $x^2+y^2\pm 2\sqrt{21}y-4=0$.

13. $x^2+y^2\pm 6x\pm 6y+9=0$. 14. $x^2+y^2-5x-7y=0$.

15. $x^2+y^2-2(4\pm\sqrt{6})(x+y)+22\pm 8\sqrt{6}=0$.

16. $x^2+y^2-4x-2y+1=0$, $x^2+y^2-20x-18y+81=0$.

17. $x^2+y^2+2x-2y-23=0$. 19. $x^2+y^2-4x-4y-66=0$.

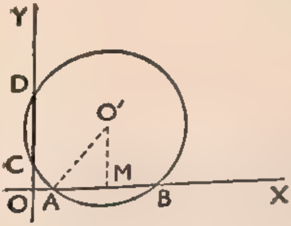
20. $2x^2+2y^2-7x+9y-6=0$. 21. (*i*) $(\frac{11}{5}, -\frac{8}{5})$; (*ii*) $(-\frac{31}{5}, -\frac{12}{5})$.

22. $x^2+y^2-4x-2y+4=0$. 23. $\frac{3\sqrt{3}}{4}(g^2+f^2-c)$.

VI-9. Length of chord :

(A) *To find the lengths of chords intercepted on the coordinate axes by the circle*

$$x^2+y^2+2gx+2fy+c=0. \quad \ldots \quad (1)$$



The equation to the axis of x is

$$y=0. \quad \ldots \quad \ldots \quad (2)$$

The points A and B are common to the circle and the x-axis.

Hence, the coordinates of A and B must satisfy both (1) and (2).

Substituting $y=0$ in (1) we have $x^2+2gx+c=0$. ... (3)
which is a quadratic equation in x to give the abscissa of the points A and B.

If $x_1\,(=OA)$ and $x_2\,(=OB)$ be the two roots of equation (3), then

$$\left.\begin{aligned} x_1+x_2&=-2g \\ x_1x_2&=\quad c \end{aligned}\right\} \quad \text{and} \quad \ldots \qquad \ldots \quad (4)$$

$$\therefore\ AB=x_2-x_1=\sqrt{(x_1+x_2)^2-4x_1x_2}$$
$$=\sqrt{4g^2-4c}=2\sqrt{g^2-c}.$$

Similarly, the length of the chord intercepted on the y-axis will be found by getting the difference of the roots of the equation $y^2+2fy+c=0$ obtained by putting $x=0$ in equation (1). The length CD will be found to be $2\sqrt{f^2-c}$.

Note : (1) If $g^2=c$, the intercept on the x-axis is zero. In this case the axis of x touches the circle.

(2) If $g^2<c$, the roots of (3) are imaginary, showing that the circle does not meet the x-axis in real points.

(3) (*i*) If c is negative, from (4) the product of the roots is negative and so one of the roots is positive and the other negative, showing that the circle meets the axis of x in two points situated on opposite sides of the origin.

(*ii*) If $c=0$, one of the roots must be zero and so the circle passes through the origin. This is also otherwise obvious from the equation.

(*iii*) If c is positive (but less than g^2), the product of the roots is positive and so the roots must be either both positive or both negative, showing that the two points of intersection must be situated on the same side of the origin.

Alternative method.

If $O'M$ be drawn perpendicular on AB from the centre O', then M must be the middle point of AB

Now $O'A=$ radius of the circle $=\sqrt{g^2+f^2-c}$

and $O'M=$ ordinate of the centre $=-f$.

Hence, $AB=2AM=2\sqrt{O'A^2-O'M^2}$

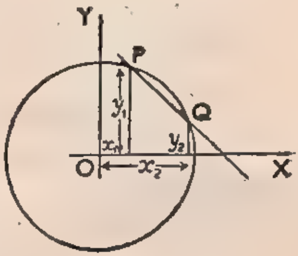
$$=2\sqrt{(g^2+f^2-c)-(-f)^2}$$
$$=2\sqrt{g^2-c}.$$

(B) *To find the length of the chord intercepted on a given straight line by a given circle.*

Let the equations of the given circle and straight line be

$$x^2+y^2=a^2 \quad \ldots \quad (1)$$

and $$y=mx+c \quad \ldots \quad (2)$$



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points of intersection of (1) and (2). Clearly the coordinates of P and Q will satisfy both the equations and hence will be obtained by solving the equations (1) and (2).

Substituting for y from (2) in (1), we have

$$x^2+(mx+c)^2=a^2$$

i.e., $$x^2(1+m^2)+2mcx+c^2-a^2=0 \quad \ldots \quad (3)$$

which is a quadratic equation in x whose roots are the abscissæ of P and Q, *viz.*, x_1 and x_2 respectively.

Hence, $$x_1+x_2=-\frac{2mc}{1+m^2}$$

and $$x_1x_2=\frac{c^2-a^2}{1+m^2}.$$

$\therefore$ $$x_2-x_1=\sqrt{(x_1+x_2)^2-4x_1x_2}$$

$$=\sqrt{\frac{4m^2c^2}{(1+m^2)^2}-\frac{4(c^2-a^2)}{(1+m^2)}}$$

$$=2\sqrt{\frac{m^2c^2-(1+m^2)(c^2-a^2)}{(1+m^2)^2}}$$

i.e., $$x_2-x_1=\frac{2}{1+m^2}\cdot\sqrt{a^2(1+m^2)-c^2} \quad \ldots \quad (4)$$

Again, since P and Q are points on the line (2), we have

$$y_1=mx_1+c \text{ and } y_2=mx_2+c$$

and so $$y_2-y_1=m(x_2-x_1).$$

$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{(x_2-x_1)^2+m^2(x_2-x_1)^2}$$

$$=(x_2-x_1)\sqrt{1+m^2}$$

i.e., $$PQ=2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}. \quad \text{from (4)}$$

Otherwise : If OM be drawn perpendicular on PQ, then the chord $PQ=2\sqrt{OP^2-OM^2}$ will give the result. The actual working out is however left as an exercise for the student.

VI-10. Equation of a chord :

To find the equation of the chord of a circle joining two given points on it.

Let the equation of the given circle be

$$x^2+y^2=a^2 \quad \cdots \quad \cdots \quad (1)$$

and let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points. (See figure of the previous article).

Now the gradient of the line $PQ=\dfrac{y_2-y_1}{x_2-x_1}$. But since (x_2, y_2) and (x_1, y_1) are points on (1), we have

$$x_2^2+y_2^2=a^2$$

and

$$x_1^2+y_1^2=a^2.$$

Hence, by subtraction, $(x_2^2-x_1^2)+(y_2^2-y_1^2)=0$

i.e., $(x_2-x_1)(x_2+x_1)+(y_2-y_1)(y_2+y_1)=0.$

$\therefore$ The gradient $\dfrac{y_2-y_1}{x_2-x_1}=-\dfrac{x_2+x_1}{y_2+y_1}.$

The required chord is therefore a line passing through $P(x_1, y_1)$ and having gradient $-\dfrac{x_2+x_1}{y_2+y_1}.$

Hence, its equation is

$$y-y_1=-\frac{x_2+x_1}{y_2+y_1}(x-x_1)$$

i.e., $(x-x_1)(x_2+x_1)+(y-y_1)(y_2+y_1)=0$

i.e., $\mathbf{x(x_1+x_2)+y(y_1+y_2)-(x_1x_2+y_1y_2+a^2)=0.}$

since $x_1^2+y_1^2=a^2.$

VI-11. Tangent at a point :

To find the equation of the tangent at the point (x_1, y_1) to the circle,

(*i*) $x^2+y^2=a^2$

(*ii*) $x^2+y^2+2gx+2fy+c=0.$

Let P be the given point (x_1, y_1). We take a point Q on the circle close to P. Let its coordinates be (x_2, y_2).

(*i*) Proceeding as in the previous article, the equation to the chord PQ is found to be

$$y - y_1 = -\frac{x_2 + x_1}{y_2 + y_1}(x - x_1).$$

If now Q tends to coincidence with P so that x_2 tends to the value x_1 and y_2 to y_1, the chord in this limiting position becomes the tangent at P. The equation of the tangent, obtained by putting $x_2 = x_1$ and $y_2 = y_1$ in the above equation, is thus

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

i.e., $\quad xx_1 + yy_1 = x_1^2 + y_1^2$

i.e., $\quad \mathbf{xx_1 + yy_1 = a^2}.$

(*ii*) The gradient of the line PQ is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

But since (x_1, y_1) and (x_2, y_2) are points on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

and $\quad x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$

Hence, by subtraction,

$$x_2^2 - x_1^2 + y_2^2 - y_1^2 + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

i.e., $\quad (x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0$

$\therefore$ the gradient $\dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}$

The *equation of the chord* PQ is therefore

$$y - y_1 = -\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}(x - x_1)$$

As before, the equation of the tangent is obtained by making $x_2 = x_1$ and $y_2 = y_1$ in the above equation. The result is

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

i.e., $\quad (x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0$

i.e., $\quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c - (gx_1 + fy_1 + c)$$

$$= -(gx_1 + fy_1 + c)$$

Since (x_1, y_1) is a point on the circle.

Hence the required equation of the tangent is

$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0}$$

Alternative Method :

(*i*) Proceeding as in *Ex. 2, Art V-3* we find the gradient of the tangent at (x_1, y_1)

$=$the gradient of the curve at the same point $=-\dfrac{x_1}{y_1}$

The required tangent is therefore the line through (x_1, y_1) having gradient $-x_1/y_1$. Its equation is thus

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

i.e., $\quad \mathbf{xx_1+yy_1=a^2}$.

(*ii*) If (x, y) be a point on the circle and $(x+\delta x, y+\delta y)$ another point close to it, then

$$(x+\delta x)^2+(y+\delta y)^2+2g(x+\delta x)+2f(y+\delta y)+c=0$$

and $\quad x^2+y^2+2gx+2fy+c=0$

$\therefore$ By Subtraction, $2x\delta x+\delta x^2+2y\delta y+\delta y^2+2g\delta x+2f\delta y=0$

i.e., $\quad \delta x(2x+2g+\delta x)+\delta y(2y+2f+\delta y)=0$

whence $\quad \dfrac{\delta y}{\delta x}=-\dfrac{2x+2g+\delta x}{2y+2f+\delta y}$

$\therefore \quad \dfrac{dy}{dx}=\lim\limits_{\delta x\to 0}\dfrac{\delta y}{\delta x}=-\dfrac{x+g}{y+f}$

Since when δx approaches zero, δy also approaches zero.

$\therefore$ The gradient of the curve at $(x_1, y_1)=-\dfrac{x_1+g}{y_1+f}$.

which is also the gradient of the tangent.

The required equation of the tangent is thus

$$y-y_1=-\frac{x_1+g}{y_1+f}(x-x_1)$$

whence the equation

$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0}$$

readily follows.

Note : The equation of the tangent (x_1, y_1) is obtained from the equation of the circle by writing xx_1 for x^2, yy_1 for y^2, $x+x_1$ for $2x$ and $y+y_1$ for $2y$. This rule will be found to apply not only to the case of the circle but to the other curves as well with which we shall have occasion to deal in subsequent chapters of this book.

VI-12. Condition for tangency :

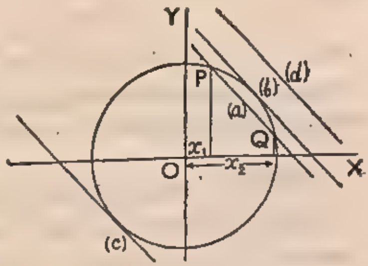
To find the condition that the straight line $y=mx+c$ *should touch the circle* $x^2+y^2=a^2$.

The points of intersection of the line $y=mx+c\ldots$ (1) and the circle $x^2+y^2=a^2\cdots$(2) are found by solving equations (1) and (2).

Thus, substituting for y from (1) in (2) we have

$$x^2+(mx+c)^2-a^2=0$$

i.e., $$x^2(1+m)^2+2mcx+c^2-a^2=0 \cdots (3)$$



which is a quadratic equation in x whose roots x_1 and x_2 are the abscissæ of the two points P and Q common to the line and the circle.

If the line is a tangent, P and Q coincide and hence $x_1=x_2$, *i.e.*, the roots of (3) are equal. This gives

$$4m^2c^2-4(1+m^2)(c^2-a^2)=0$$

or, $$c^2=a^2(1+m^2)$$

i.e., $$\mathbf{c=\pm a\sqrt{1+m^2}}. \quad \ldots \quad \ldots \quad (4)$$

This then is the required condition.

Remark : The roots of the equation (3) are imaginary

if $4m^2c^2-4(1+m^2)(c^2-a^2)<0$

i.e., if $a^2(1+m^2)<c^2$

i.e., if $c^2>a^2(1+m^2)$.

In this case the line meets the circle in imaginary points, *i.e.*, geometrically it does not meet the circle at all. The line (*d*) in the figure corresponds to this case.

The roots of the equation (3) are real and different if $c^2<a^2(1+m^2)$. In this case the line meets the circle in two real points as in the case of the line (a) in the figure.

Alternative method :

The condition for tangency may be easily deduced from the geometrical property of the tangent, *viz*, that it is perpendicular to the radius through the point of contact.

The line $y=mx+c$ will touch the circle $x^2+y^2=a^2$ if the

length of the perpendicular from the centre (0, 0) upon it be equal to the radius a

i.e., if $\pm\dfrac{c}{\sqrt{1+m^2}}=a$

i.e., if $c=\pm a\sqrt{1+m^2}$.

Note. This method being based on the property of the tangent to a circle, is not applicable to any other curve.

VI-13. Tangents in a given direction :

If the gradient m of a line is given, we find there are two values of c, *viz.*, those obtained in the previous article which will make $y=mx+c$ a tangent to the circle $x^2+y^2=a^2$. Hence, corresponding to a given gradient m there are always two parallel tangents to the circle $x^2+y^2=a^2$ whose equations are

$$\mathbf{y=mx\pm a\sqrt{1+m^2}}.$$

In the figure of the last article, (b) and (c) are the two parallel tangents.

VI-14. Number of tangents from a point :

To show that two tangents (real or imaginary) can always be drawn from a point to a circle.

Let the equation to the circle be $x^2+y^2=a^2$ and let (x_1, y_1) be the given point.

The line $y=mx+a\sqrt{1+m^2}$ ... .. (1)

is always a tangent to the circle $x^2+y^2=a^2$ and for different values of m it represents tangents in different directions. We are required to prove that two of these pass through the given point (x_1, y_1).

Now, if (1) passes through (x_1, y_1)

we have $y_1=mx_1+a\sqrt{1+m^2}$

or, $(y_1-mx_1)^2=a^2(1+m^2)$

i.e., $m^2(x_1{}^2-a^2)-2x_1y_1m+y_1{}^2-a^2=0.$ ... (2)

The above equation is quadratic in m and therefore gives two values of m corresponding to each of which we have a tangent passing through (x_1, y_1).

Hence, two tangents can always be drawn from the given point to the circle and these are

$$\mathbf{y-y_1=m_1(x-x_1)}$$

and $$\mathbf{y-y_1=m_2(x-x_1)}$$

where m_1 and m_2 are the two roots of equation (2).

The discriminant of equation (2)

$$=4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2)$$
$$=4a^2(x_1^2+y_1^2-a^2)$$
$$=4a^2(r^2-a^2)$$

where r is the distance of the given point from the centre of the circle.

Now,

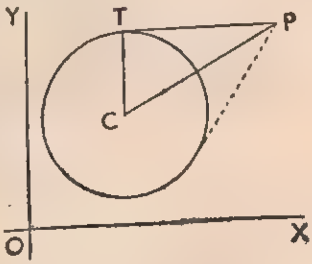
(*i*) if the disc. is positive, *i.e.*, $r>a$, *i.e.*, the point lies outside the circle, we get two real and distinct roots and hence, two distinct tangents can be drawn ;

(*ii*) if the disc. is zero, *i.e.*, $r=a$, *i.e*, the point lies on the circle, we get two equal roots and hence, the two tangents become coincident ;

(*iii*) if the disc. is negative, *i.e.*, $r<a$, *i.e.*, the point lies inside the circle, we get two imaginary roots and in this case both the tangents are imaginary, *i.e.*, geometrically no tangents can be drawn.

VI-15. Length of the tangent :

To find the length of the tangent drawn from an external point (x_1, y_1) to the circle $(x-\alpha)^2+(y-\beta)^2=r^2$.



Let P be the external point, PT the tangent, T being the point of contact and C the centre of the circle.

Now, the centre C is the point (α, β).

$\therefore\ PC^2=(x_1-\alpha)^2+(y_1-\beta)^2$.

Also $CT=$ radius $=r$.

Then since, $\angle CTP$ is a right ang'e, we have

$$PT^2=PC^2-CT^2$$

i.e., $$\mathbf{PT^2=(x_1-\alpha)^2+(y_1-\beta)^2-r^2}.$$

In a similar manner the square of the length of the tangent from the point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ will be found to be

$$\mathbf{x_1^2+y_1^2+2gx_1+2fy_1+c}.$$

The square of the tangent from an external point to a circle is therefore obtained by substituting the coordinates of the point for x and y in the left-hand member of the equation to the circle, the equation being so written that the right-hand member is zero and the coefficients of x^2 and y^2 are each unity.

Cor. The square of the length of the tangent from the origin (0, 0) to the circle $x^2+y^2+2gx+2fy+c=0$ is c. The result can also be arrived at independently thus :

$$\begin{aligned}\text{Square of the tangent} &= OC^2-(\text{radius})^2\\ &= g^2+f^2-(g^2+f^2-c)\\ &= c\end{aligned}$$

since the centre is $(-g, -f)$ and the radius is $\sqrt{g^2+f^2-c}$.

VI-16. Normal at a point :

To find the equation to the normal to

(*i*) *the circle* $x^2+y^2=a^2$

(*ii*) *the circle* $x^2+y^2+2gx+2fy+c=0$

at the point (x_1, y_1).

(*i*) The equation of the tangent at (x_1, y_1) is

$$xx_1+yy_1=a^2$$

i.e.,
$$y=-\frac{x_1}{y_1}x+\frac{a^2}{y_1} \quad \ldots\ldots (1)$$

$\therefore$ The equation to the normal which is a straight line through (x_1, y_1) prependicular to the tangent (1) is

$$y-y_1=m(x-x_1) \quad \ldots\ldots (2)$$

where
$$m\times\left(-\frac{x_1}{y_1}\right)=-1$$

i.e.,
$$m=\frac{y_1}{x_1}.$$

Hence, substituting for m in (2), the required equation is

$$y-y_1=\frac{y_1}{x_1}(x-x_1)$$

i.e., $$\mathbf{xy_1 - yx_1 = 0.}$$

(*ii*) The equation of the tangent at (x_1, y_1) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

i.e., $$x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0$$

i.e., $$y = -\frac{x_1 + g}{y_1 + f}x - \frac{gx_1 + fy_1 + c}{y_1 + f}.$$

The equation to the normal which is a straight line through (x_1, y_1) perpendicular to the tangent is

$$y - y_1 = m(x - x_1)$$

where $$m \times \left(-\frac{x_1 + g}{y_1 + f}\right) = -1$$

i.e., $$m = \frac{y_1 + f}{x_1 + g}.$$

Hence, the equation to the normal is

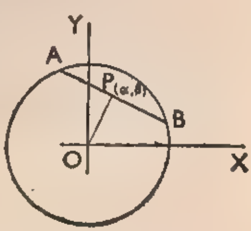
$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

i.e., $$\mathbf{x(y_1 + f) - y(x_1 + g) - fx_1 + gy_1 = 0.}$$

Alternative method : Since from Geometry, the radius through the point of contact is perpendicular to the tangent, in (*i*) the line joining the centre (0, 0) and the point (x_1, y_1), and in (*ii*) the line joining the centre $(-g, -f)$ and the point (x_1, y_1) must be the normal. The equations may easily be derived.

VI-17. Chord having its middle point given :

To find the equation of the chord of the circle $x^2 + y^2 = a^2$ which is bisected at a given point (α, β).



Let $P(\alpha, \beta)$ be the given point. Then if AB be the required chord, it must be one passing through P and be perpendicular to OP.

Now, the gradient of OP is $\frac{\beta}{\alpha}$.

Hence, the gradient of AB is $-\frac{\alpha}{\beta}$. [*Art. IV*-10 (2)]

Therefore, the required equation of the chord AB is

$$y - \beta = -\frac{\alpha}{\beta}(x - \alpha)$$

i.e., $$\mathbf{(x - \alpha)\alpha + (y - \beta)\beta = 0.}$$

VI-18. Common chord :

To find the equation of the common chord of two given circles.

Let the equations of the given circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad \cdots \quad (1)$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad \cdots \quad (2)$$

Consider the equation

$$(x^2+y^2+2g_1x+2f_1y+c_1)-(x^2+y^2+2g_2x+2f_2y+c_2)=0. \qquad \cdots \quad (3)$$

Let the two circles (1) and (2) intersect in P and Q. Then the coordinates of P and Q must satisfy both the equations (1) and (2). Hence, these coordinates also satisfy equation (3).

Equation (3) therefore represents some locus through P and Q.

Also equation (3) when simplified reduces to

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0 \quad \ldots \quad (4)$$

which being an equation of the first degree in x and y represents a straight line.

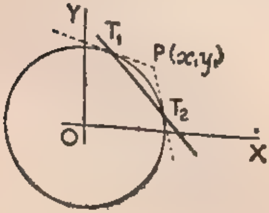
Hence, $\mathbf{2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0}$

is the equation of the straight line passing through the points of intersection of the two given circles, that is, the equation of the common chord.

Note : Even if the circles do not geometrically intersect, the locus represented by the above equation is a real straight line. The geometrical interpretation of the straight line in this case is however beyond the scope of this elementary volume.

VI-19. Chord of contact :

Def. The chord joining the points of contact of the two tangents that can be drawn from an external point to a circle is called the chord of contact of tangents from the point.



To find the equation of the chord of contact of tangents drawn from the external point (x_1, y_1) to the circle $x^2+y^2=a^2$.

Let $T_1(\alpha_1, \beta_1)$ and $T_2 (\alpha_2, \beta_2)$ be the points of contact of tangents from the given point $P(x_1, y_1)$ to the given circle.

The tangents at T_1 and T_2 are respectively

$$x\alpha_1 + y\beta_1 = a^2$$

and $$x\alpha_2 + y\beta_2 = a^2.$$

Since these tangents pass through P, its coordinates x_1 and y_1 must satisfy both the equations.

Hence, $$x_1\alpha_1 + y_1\beta_1 = a^2 \quad \cdots (1)$$

and $$x_1\alpha_2 + y_1\beta_2 = a^2. \quad \cdots (2)$$

The equation to the chord of contact T_1T_2 is therefore given by $$\mathbf{xx_1 + yy_1 = a^2} \quad \ldots (3)$$

for, (*i*) the points $T_1(\alpha_1, \beta_1)$ and $T_2(\alpha_2, \beta_2)$ lie on the locus given by (3) which follows from the relations (1) and (2); and (*ii*) being of the first degree in x and y, the locus represented by (3) is a straight line.

Hence, (3) being a straight line passing through T_1 and T_2 must be the required equation of the chord of contact.

If the equation of the circle be given in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

then proceeding in the same way it will be found that the equation of the chord of contact is

$$\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.}$$

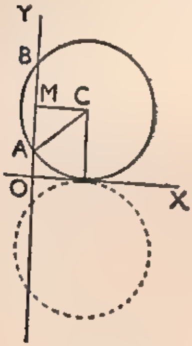
Note : 1. It will be seen that the equation of the chord of contact of tangents drawn from (x_1, y_1) is the same as the equation of the tangent at (x_1, y_1). It should however be remembered that in the case of the chord of contact (x_1, y_1) is an external point whereas in the case of the tangent it is a point on the circle.

Note : 2. The equations being identical the rule to write down the equation of the tangent applies equally to that of the chord of contact.

WORKED OUT EXAMPLES

Ex. 1. *Find the equation to the circle which touches the axis of x at the point* (3, 0) *and cuts off from the axis of y an intercept of length* 8.

If r be the radius of the circle, then the centre C is clearly the point $(3, r)$ [See *figure*]



If CM be drawn perpendicular to OY, then

$$AM = \sqrt{CA^2 - CM^2}$$
$$= \sqrt{r^2 - 3^2}.$$

Hence, $8 = \text{intercept } AB$

$$= 2AM = 2\sqrt{r^2 - 3^2}.$$

$\therefore$ $16 = r^2 - 9$ whence $r = 5$.

Hence, the centre of the circle is the point $(3, 5)$ and the radius is 5.

The required equation is then

$$(x-3)^2 + (y-5)^2 = 5^2$$

i.e., $$x^2 + y^2 - 6x - 10y + 9 = 0$$

Obviously, there is another circle also satisfying the given conditions. The centre of this circle is $(3, -5)$ and its equation is $x^2 + y^2 - 6x + 10y + 9 = 0$.

Otherwise :

Let the required equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \dots \quad (1)$$

It meets the axis of x (*i.e.*, the line $y = 0$) in points given by

$$x^2 + 2gx + c = 0. \quad \dots \quad (2)$$

Since the axis of x touches the circle at the point $(3, 0)$ the two roots of (2) must be equal and each equal to 3, so that (2) must be identical with $(x-3)^2 = 0$

i.e., with $x^2 - 6x + 9 = 0$.

Hence, $g = -3$ and $c = 9$.

The equation (1) then reduces to

$$x^2 + y^2 - 6x + 2fy + 9 = 0 \quad \dots \quad (3)$$

The circle (3) meets the axis of y (*i.e.*, the line $x = 0$) in points given by

$$y^2 + 2fy + 9 = 0.$$

If y_1 and y_2 be the roots of this equation, then by the given condition,

$$8=y_1 \sim y_2=\sqrt{(y_1+y_2)^2-4y_1y_2}=\sqrt{4f^2-36}$$

$\therefore \quad f^2=25$ whence $f=\pm 5$.

Hence, the required equations are $x^2+y^2-6x\pm 10y+9=0$.

Ex. 2. *Find the equations of the tangents of the circle $x^2+y^2+4x-2y-4=0$ which are perpendicular to the straight line $3x+4y+5=0$.*

The equation to the circle can be written in the form

$$(x+2)^2+(y-1)^2=3^2. \qquad \dots \quad (1)$$

The centre of the circle is therefore the point $(-2, 1)$ and the radius is 3.

Any straight line perpendicular to the line $3x+4y+5=0$ is

$$4x-3y+c=0. \qquad \dots \quad (2)$$

If the line (2) is a tangent to the circle (1) its distance from the centre $(-2, 1)$ must be equal to $\pm$ 3.

Hence, $\dfrac{-8-3+c}{\sqrt{4^2+3^2}}=\pm 3$

i.e., $c=11\pm 15=26$ or, -4.

The required equations are therefore

$$4x-3y+26=0 \text{ and } 4x-3y-4=0.$$

Ex. 3. *Prove that the line $3x-4y=5$ touches the circle $x^2+y^2=1$ and find the point of contact.*

The given line is $y=\dfrac{3x-5}{4}$.

It meets the circle $x^2+y^2=1$ in points whose abscissæ are given by

$$x^2+\left(\frac{3x-5}{4}\right)^2=1$$

i.e., $25x^2-30x+9=0$

i.e., $(5x-3)^2=0$

$\therefore \quad x=\frac{3}{5}, \frac{3}{5}$.

The roots being equal, the two points of intersection are coincident and hence the line is a tangent.

Putting $x=\frac{3}{5}$ in the equation to the straight line,

$$y=\frac{\frac{9}{5}-5}{4}=-\tfrac{4}{5}.$$

Hence, the point of contact is $(\frac{3}{5}, \ -\frac{4}{5})$.

Ex. 4. *Find the condition that the line $lx+my+n=0$ should be (i) a tangent (ii) a normal to the circle $x^2+y^2+2gx+2fy+c=0$.*

(*i*) The centre of the circle is $(-g, -f)$ and the radius is $\sqrt{g^2+f^2-c}$.

If the line $lx+my+n=0$ be a tangent then the length of the perpendicular from the centre upon it must be equal to the radius

$$i.e., \quad \frac{-gl-fm+n}{\sqrt{l^2+m^2}}=\sqrt{g^2+f^2-c}$$

$$\text{or,} \quad (gl+fm-n)^2=(l^2+m^2)(g^2+\quad -c$$

which reduces to

$$c(l^2+m^2)+n^2-(fl-gm)^2-2fmn-2gnl=0.$$

This then is the required condition.

(*ii*) Since, the normal to a circle at any point of it always passes through the centre, we have

$$l(-g)+m(-f)+n=0$$

$$\text{or,} \quad gl+fm-n=0$$

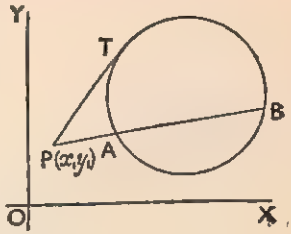
which is therefore the required condition.

Ex. 5. *From a point on a chord of a circle produced a tangent is drawn to the circle. Prove that the rectangle contained by the segments of the chord is equal to the square on the tangent.*

Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0$$

and let $P(x_1, y_1)$ be a point on the chord AB produced. From P a tangent PT is drawn to the circle.



If AB makes an angle θ with the x-axis, then its equations when expressed in symmetrical form are

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

[*Art. IV-3 (B)*]

Now, the coordinates (x, y) of any point on the line at a distance r from (x_1, y_1) are given by

$$x=x_1+r\cos\theta,\quad y=y_1+r\sin\theta.$$

If this point is on the circle, we have

$$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2+2g(x_1+r\cos\theta)$$
$$+2f(y_1+r\sin\theta)+c=0$$

i.e., $$r^2(\cos^2\theta+\sin^2\theta)+2r\{(x_1+g)\cos\theta+(y_1+f)\sin\theta\}$$
$$+x_1^2+y_1^2+2gx_1+2fy_1+c=0$$

which is a quadratic equation in r, the roots of which give the distances of the points of intersection of the line and the circle from (x_1, y_1) *i.e.,* PA and PB.

Now, the product of the roots of the equation is

$$x_1^2+y_1^2+2gx_1+2fy_1+c \quad (\because \cos^2\theta+\sin^2\theta=1)$$

which is the expression for PT^2.

Hence, $PA.PB=PT^2$

which proves the proposition.

EXERCISE VI(B)

1. Find the equation to the circle which touches the x-axis at a distance $+5$ from the origin and cuts off a chord of length 24 from the y-axis. [*C. U.*]

2. Find the length of the chord cut off

 (*i*) from the line $3x-4y+15=0$ by the circle $x^2+y^2=25$.

 (*ii*) from the line $y=2x-5$ by the circle $x^2+y^2-6x+8y-5=0$.

3. Find the middle point of the chord cut off by the circle $x^2+y^2=a^2$ from the line $y=mx+c$.

4. If $P(\alpha, \beta)$ be a point within the circle $x^2+y^2=a^2$, find the length of and the equation to the least chord of it passing through P.

5. Find the equation to the common chord of the circles

(i) $x^2+y^2+2x+6y-6=0$ and $x^2+y^2-4x-6y-12=0$,

(ii) $2x^2+2y^2-3x-5y-5=0$ and $x^2+y^2+2x+3y=0$

and prove that it is perpendicular to the line joining the centres of the circles.

6. Find where the line $3x+4y+7=0$ cuts the circle $x^2+y^2-4x-6y-12=0$. [C. U.]

7. Find the equation to the chord of the circle $x^2+y^2=81$ which is bisected at the point $(-2, 3)$. [C. U.]

8. Find the coordinates of the point of contact of the tangent $y=mx+a\sqrt{1+m^2}$ to the circle $x^2+y^2=a^2$. [C. U.]

9. Find the equations of the tangents to the circle $x^2+y^2=9$ which are parallel to the line $3x+4y=0$. [C. U.]

10. Find the equations of the tangents to the circle $x^2+y^2=a^2$ which are perpendicular to the straight line $y=mx+c$.

11. Find the equations of the tangents to the circle $x^2+y^2-6x+4y=12$, which are parallel to the straight line $4x+3y+5=0$. [C. U.]

12. Find the equations of the tangents to the circle $x^2+y^2-6x+4y-7=0$ which are perpendicular to the straight line $y=2x+3$.

13. Prove that the straight line $x-\sqrt{3y}+2=0$ touches the circle $x^2+y^2=1$, and determine the point of contact.

14. Prove that the straight line $y=x+3\sqrt{2}$ touches the circle $x^2+y^2=9$, and find the point of contact.

15. Prove that the straight line $x+y=2+\sqrt{2}$ touches the circle $x^2+y^2-2x-2y+1=0$, and find the point of contact.

16. Prove that the lines $x=7$ and $y=8$ both touch the circle $x^2+y^2-4x-6y=12$. Find the points of contact. [C. U. 1955]

17. Find the equations of the two tangents to the circle $x^2+y^2=3$ which make an angle 60° with the axis of x. [C. U. 1955]

18. Find the equations of the tangents to the circle $x^2+y^2=1$ which pass through the point $(0, 2)$.

19. Find the condition that the line $Ax+By+C=0$ may touch the circle

(i) $x^2+y^2=a^2$;

(ii) $(x-\alpha)^2+(y-\beta)^2=a^2$.

20. Find the points on the circle $x^2+y^2+2x-6y-22=0$ at which the tangents are parallel to the straight line $x+y=0$.

21. Find the points on the circle $x^2+y^2-14x-10y+45=0$ at which the tangents are perpendicular to the straight line $2x-5y+4=0$.

22. Find the equation to the circle touching the coordinate axes and also the line $x+y=5$, the centre being in the positive quadrant.

23. Find the length of the tangent from

(i) the point (6, 5) to the circle $x^2+y^2-6x-8y+24=0$;

(ii) the point (5, 4) to the circle $3x^2+3y^2-4x-2y-20=0$;

(iii) any point on the circle $x^2+y^2=a^2$ to the circle $x^2+y^2=a'^2$ $(a > a')$.

24. The length of the tangent from (f, g) to the circle $x^2+y^2=6$ is twice the length of the tangent to the circle $x^2+y^2+3x+3y=0$; show that $f^2+g^2+4f+4g+2=0$. [C. U.]

25. Prove that the lengths of the tangents drawn from any point on the straight line $6x+4y+3=0$ to the circles $x^2+y^2-4x-6y+10=0$ and $x^2+y^2+8x+2y+16=0$ are equal.

26. Find the equations of the tangents to the circle $x^2+y^2=41$ at the points where $x=5$.

27. Find the equations of the tangents to the circle $x^2+y^2=5$ at the points where the straight line $x-y-3=0$ meets the curve.

28. Write down the equation of the tangent to the circle

(i) $x^2+y^2-3x+4y=0$, at the origin;

(ii) $x^2+y^2-8x+2y-8=0$, at the point $(7, -5)$.

29. Find the equation of the tangent and the normal to the circle

(i) $x^2+y^2-14x+8y+40=0$, at the point $(3, -1)$;

(ii) $x^2+y^2+3x-4y-6=0$, at the point $(-5, 2)$.

30. If the tangents at (x_1, y_1) and (x_2, y_2) on the circle $x^2+y^2+2gx+2fy+c=0$ are perpendicular, show that

$$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+g^2+f^2=0.$$ [C. U.]

31. Show that the equation of the tangent at the point (x_1, y_1) to the circle $(x-\alpha)^2+(y-\beta)^2=r^2$, may be written as

$$(x-x_1)(x_1-\alpha)+(y-y_1)(y_1-\beta)=0.$$ [C. U.]

32. Prove that if $y=x \sin \alpha + a \sec \alpha$ be a tangent to the circle $x^2+y^2=a^2$, then $\cos^2 \alpha=1$. [C. U.]

33. Show that the circle $x^2+y^2-2ax-2ay+a^2=0$ touches the axes of x and y; and find the chord of contact. [C. U.]

34. Find the condition that the chord of contact of tangents from the point (x_1, y_1) to the circle $x^2+y^2=a^2$ should subtend a right angle at the centre. [C. U.]

[**Hints :** The figure formed by the tangents and the radii through the points of contact is a square.]

35. Find the coordinates of the point of intersection of tangents to the circle $x^2+y^2=3$ at its points of intersection with the line $7x-6y=9$.

Answers :

1. $x^2+y^2-10x\pm 26y+25=0$. 2. (*i*) 8 ; (*ii*) 10.

3. $\left(-\frac{mc}{1+m^2}, \frac{c}{1+m^2}\right)$. 4. $2\sqrt{-(a^2+\beta^2-a^2)}$; $ax+\beta y=a^2+\beta^2$.

5. (*i*) $x+2y+1=0$; (*ii*) $7x+11y+5=0$.

6. The line cuts the circle in two coincident points (*i.e.*, touches it) at the point $(-1, -1)$.

7. $2x-3y+13=0$. 8. $\left(-\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$. 9. $3x+4y\pm 15=0$.

10. $my+x=\pm a\sqrt{1+m^2}$. 11. $4x+3y+19=0$, $4x+3y-31=0$.

12. $x+2y+11=0$, $x+2y-9=0$. 13. $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. 14. $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$.

15. $\left(\frac{\sqrt{2}+1}{\sqrt{2}}, \frac{\sqrt{2}+1}{\sqrt{2}}\right)$. 16. $(7, 3)$; $(2, 8)$. 17. $y=\sqrt{3}x\pm 2\sqrt{3}$.

18. $y=\pm\sqrt{3}x+2$. 19. (*i*) $C=\pm a\sqrt{A^2+B^2}$; (*ii*) $A\alpha+B\beta+C=\pm a\sqrt{A^2+B^2}$ 20. $(3, 7)$, $(-5, -1)$. 21. $(2, 3)$, $(12, 7)$.

22. $x^2+y^2-2a(x+y)+a^2=0$ where $a=\frac{5}{2}(2\pm\sqrt{2})$.

23. (*i*) 3 ; (*ii*) 5 ; (*iii*) $\sqrt{a^2-a'^2}$. 26. $5x\pm 4y=41$.

27. $x-2y=5$, $2x-y=5$. 28. (*i*) $3x-4y=0$; (*ii*) $3x-4y=41$.

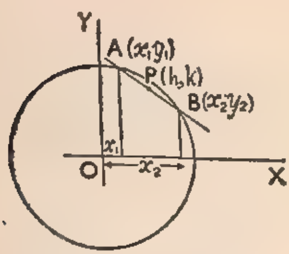
29. (*i*) $4x-3y=15$, $3x+4y=5$; (*ii*) $x+5=0$, $y-2=0$.

33. length of the chord $=a\sqrt{2}$, equation to the chord : $x+y=a$.

34. $x_1^2+y_1^2=2a^2$. 35. $(\frac{7}{3}, -2)$.

VI–20. Locus problems :

Problem 1. *To prove that the locus of the middle points of a system of parallel chords of a circle is a straight line passing through the centre (diameter) perpendicular to the chords.*



Let the equation of the circle be $x^2+y^2=a^2$ ...(1) and let m be the gradient of the system of parallel chords so that the equation of any one of the system of chords, say AB, is given by

$$y=mx+c \qquad \cdots (2)$$

where different values of c give different chords of the system. [See Art. IV-8 (1)]

If $A(x_1, y_1)$ and $B(x_2, y_2)$ be the points in which the line (2) meets the circle (1) then x_1 and x_2 must be the roots of the equation

$$x^2+(mx+c)^2=a^2$$

i.e., $$x^2(1+m^2)+2mcx+c^2-a^2=0.$$

If (h, k) be the coordinates of P the middle point of AB,

then $$h=\frac{x_1+x_2}{2}=-\frac{mc}{1+m^2}. \quad \dots \quad \dots \quad (3)$$

Similarly, y_1 and y_2 are the roots of the quadratic in y obtained by eliminating x between equations (1) and (2), *viz.*,

$$\left(\frac{y-c}{m}\right)^2+y^2=a^2,$$

i.e., $$y^2(1+m^2)-2cy+c^2-a^2m^2=0.$$

$$\therefore \quad k=\frac{y_1+y_2}{2}=\frac{c}{1+m^2}. \quad \dots \quad \dots \quad (4)$$

Now, to get the locus of P we require a relation between its coordinates h and k which must not contain c in order that it may hold for any value of c.

Dividing (4) by (3), we have

$\frac{k}{h}=-\frac{1}{m}$, a relation independent of c.

$\therefore$ Substituting current coordinates, the required locus is given by

$$\frac{y}{x}=-\frac{1}{m} \qquad i.e. \quad y=-\frac{1}{m}x$$

which is clearly a line hrough the centre (0, 0) perpendicular to the chords.

Problem 2. *To find the locus of a point which moves so that the tangents drawn from it to a given circle are always at right angles.*

Let the given circle be $x^2+y^2=a^2$. ... (1)

If $P(h, k)$ be a point on the locus then the two tangents to (1) which pass through P must be at right angles.

Now, any tangent to (1) is given by

$$y=mx+a\sqrt{1+m^2}$$

If it passes through $P(h, k)$, then

$$k=mh+a\sqrt{1+m^2},$$

i.e., $$(k-mh)^2=a^2(1-m^2),$$

i.e., $$m^2(h^2-a^2)-2mhk+k^2-a^2=0. \quad \cdots \quad (2)$$

The roots m_1 and m_2 of the last equation give the gradients of the two tangents which pass through P.

Since, the tangents are at right angles, we have

$$m_1m_2=-1.$$

Hence, from equation (2),

$$\frac{k^2-a^2}{h^2-a^2}=-1,$$

i.e., $$h^2+k^2=2a^2.$$

This being the relation between the coordinates of any point on the locus, we have on substitution of current coordinates,

$$\mathbf{x^2+y^2=2a^2}$$

as the required equation to the locus. The locus is clearly a **concentric circle.**

Problem 3. *To find the locus of a point which moves so that the lengths of the tangents drawn from it to two given circles are always equal.*

Let the given circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \cdots \quad (1)$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad \cdots \quad (2)$$

Let $P(h, k)$ be any point to the locus. Then, if PT, PT' be the tangents from P to the circles (1) and (2) the condition to be satisfied by P is

$$PT=PT'$$

i.e., $$PT^2=PT'^2$$

Hence, $$h^2+k^2+2g_1h+2f_1k+c_1$$
$$=h^2+k^2+2g_2h+2f_2k+c_2 \quad \cdots [Art.\ VI\text{-}15]$$

i.e., $$2(g_1-g_2)h+2(f_1-f_2)k+c_1-c_2=0.$$

$\therefore$ The locus of (h, k) is given by

$$\mathbf{2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0}$$

which is clearly a straight line.

[The straight line is called the **Radical axis** of the two circles. It is also the common chord when the circles intersect. (See *Art. VI* 18.)]

Problem 4. *To find the locus of the middle points of chords of the circle* $x^2+y^2=a^2$ *which pass through a given point* (α, β).

Let (h, k) be a point on the locus. Then the chord of the circle $x^2+y^2=a^2$ whose middle point is (h,k) must pass through the given point (α, β).

Now the chord whose middle point is (h, k) is given by

$$(x-h)h+(y-k)k=0. \qquad [\ Art.\ VI\text{-}17\ ]$$

If it passes through (α, β), we have

$$(\alpha-h)h+(\beta-k)k=0,$$

i.e., $$h^2+k^2-\alpha h-\beta k=0.$$

Hence, the locus of (h, k) is given by

$$\mathbf{x^2+y^2-\alpha x-\beta y=0}$$

which is clearly a circle.

[**Note :** The locus is independent of the radius of the circle.]

EXERCISE VI(C)

1. $A(-a, 0)$ and $B(a, 0)$ are two fixed points. A point P moves so that PA^2+PB^2 is constant equal to $2k^2$ $(k>a)$; find the locus of P.

Discuss the case when $k=a$.

2. A point moves so that the sum of the squares of its distances from the angular points of a triangle is constant ; prove that its locus is a circle. [*C. U.* 1952]

What can you say about the centre of the circle ?

3. A point moves so that the sum of the squares of its distances from the four sides of a square is constant. Prove that it always lies on a circle. [*C. U.*]

4. A straight line of constant length $2l$ slides having its extremities on the axes of coordinates. Find the locus of the middle point of the line.

5. A chord of constant length $2l$ is made to slide within the circle $x^2+y^2=a^2$. Find the locus of the middle point of the chord.

6. The base of a triangle is fixed and its vertex moves so that the ratio of the sides is constant. Prove that the locus of the vertex is a circle.

[**Hints :** Take the middle point of the base as origin, the base as the axis of x and a line through the middle point perpendicular to the base as the axis of y.]

7. A point moves so that the square of its distance from the origin is $2a$ times its distance from the straight line $x+\frac{3}{2}a=0$. Show that the locus of the point is a circle and find its centre and radius.

What is the locus when the line is $x=\frac{1}{2}a$?

8. Prove that the locus of a point whose distance from a fixed point is in a constant ratio to the tangent drawn from it to the circle $x^2+y^2+2gx+2fy+c=0$, is a circle.

[C. U.]

9. A point moves so that the tangents from it to two given circles are in a constant ratio. Prove that the locus of the point is a circle.

10. If a chord of the circle $x^2+y^2=a^2$ always subtends a right angle at the centre, find the locus of its middle point.

[**Hints :** The length of a chord having (h, k) its middle point is $2\sqrt{a^2-(h^2+k^2)}$]

11. Prove that the locus of the feet of the perpendiculars from the origin on chords of the circle $x^2+y^2=a^2$, which pass through a fixed point (α, β) is a circle independent of the radius of the given circle.

[**Hints :** If (h, k) be a point on the locus, then the line through (h, k) perpendicular to the line joining it to the origin must pass through the fixed point (α, β)].

12. Find the locus of the middle points of chords of a circle which pass through a fixed point on the circumference.

13. Show that the locus of the feet of the perpendiculars from a fixed point on a circle upon its diameters is another circle, and determine its centre and radius.

[C. U.]

14. If the tangent from a point P to the circle $x^2+y^2=a^2$ be equal to the perpendicular from P to the straight line $lx+my+n=0$, find the locus of P.

[C. U.]

15. Whatever be the value of α, prove that the locus of the intersection of the straight lines $x \cos \alpha+y \sin \alpha=a$ and $x \sin \alpha-y \cos \alpha=b$, is a circle.

[C. U. 1951]

Answers :

1. $x^2+y^2=k^2-a^2$; when $k=a$, the locus is a point-circle at the origin.
2. The centre of the circle is the centroid of the triangle.
4. $x^2+y^2=l^2$.
5. $x^2+y^2=a^2-l^2$.
7. Centre $(a, 0)$, radius $=2a$; a point-circle at $(a, 0)$.
10. $x^2+y^2=\frac{1}{2}a^2$.
11. The locus is $x^2+y^2-\alpha x-\beta y=0$ which is independent of a.
12. The locus is a circle described on the radius through the given point as diameter.
13. If the equation to the circle be $x^2+y^2=a^2$ and (α, β) be the fixed point, then the centre of the locus is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ and the radius is $\frac{a}{2}$.
14. $(l^2+m^2)(x^2+y^2-a^2)=(lx+my+n)^2$.
15. $x^2+y^2=a^2+b^2$.

PART TWO

CONIC SECTIONS

CHAPTER VII

THE PARABOLA

VII·1. Definitions

A **Conic Section** or simply a **conic** is the locus of a point which moves in a plane so that its distance from a fixed point in the plane always bears a constant ratio to its distance from a fixed straight line in the same plane.

The fixed point is called the **Focus,** the fixed straight line is called the **Directrix** and the constant ratio is called the **Eccentricity** usually denoted by the letter **e**.

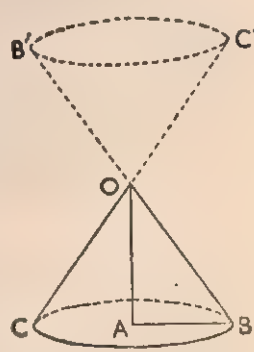
It is the magnitude of this ratio e on which depends the shape of the curve traced by the moving point, and the curve is called

A parabola   *if* $e=1$

An ellipse   *if* $e<1$

and   *A hyperbola if* $e>1$.

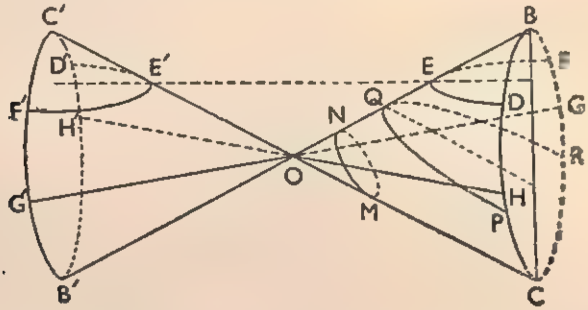
Sections of a Cone :



If a right-angled triangle OBA is rotated about one of its sides, say OA, containing the right angle, the surface generated by the hypotenuse OB is a **Right Circular Cone.** The point O is the **Vertex,** OA is the **Axis** and the angle AOB is the semi-vertical angle of the Cone. The point B describes a circle known as the **Base** of the Cone. The line joining the vertex O to any point on the rim of the base is a **Generator** of the Cone. If the generators be produced beyond the vertex an equal and opposite cone is formed and the whole surface is called a **Double Cone.**

If a plane intersects this surface the line of section will be a curve the shape of which will depend on the position of the intersecting plane.

(1) If the plane be at right angles to the axis of the Cone, the section is clearly a **Circle** such as BC of the figure.



(2) If we take an oblique section which intersects both the generating lines OB and OC, the resulting curve MN is a closed one called an **Ellipse.**

(3) When the plane is parallel to a generator, we get a curve which extends beyond limit as the generators of the Cone are produced. The curve is a **Parabola**—as PQR in the figure.

(4) When the plane intersects both the parts of the Cone we get a curve called a **Hyperbola** consisting of two branches DEF, $D'E'F'$ in the figure, the branches spreading out continuously in opposite directions.

(5) A plane passing through the axis of the Cone gives a pair of intersecting straight lines as GOG', HOH'.

The curves parabola, ellipse and hyperbola are called Conic Sections because they are obtained as sections of a right circular cone in different positions. Although a circle and a pair of intersecting straight lines are strictly speaking Conic Sections, they are not generally so called and the name Conic is confined only to the curves parabola, ellipse and hyperbola.

The eccentricity of a parabola being unity, we can define it as follows :

A **Parabola** is the locus of a point which moves in a plane so that its distance from a fixed point in the plane is always equal to its distance from a fixed straight line in the same plane.

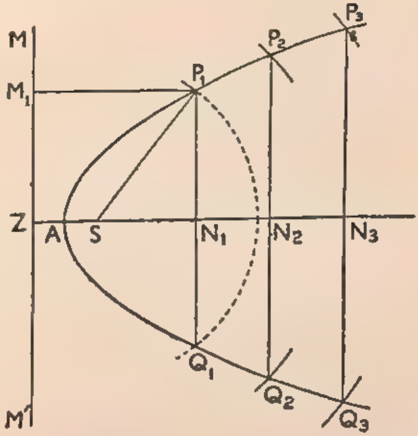
The fixed point is called the **Focus** and the fixed straight line is called the **Directrix.**

VII-2. Construction of the Parabola :

To trace the parabola when the directrix and the position of the focus are given.

Let S be the focus and MM' the directrix.

Draw SZ perpendicular to MM' and bisect SZ at A.



Then since $SA = AZ$, A is a point on the parabola.

Take a point N_1 on AS or AS produced and draw $P_1N_1Q_1$ perpendicular to ZN_1. With centre S and radius ZN_1 draw an arc cutting $P_1N_1Q_1$ at P_1 and Q_1. Then P_1 and Q_1 are points on the parabola, for, joining SP_1 and drawing P_1M_1 perpendicular to MM', we have clearly

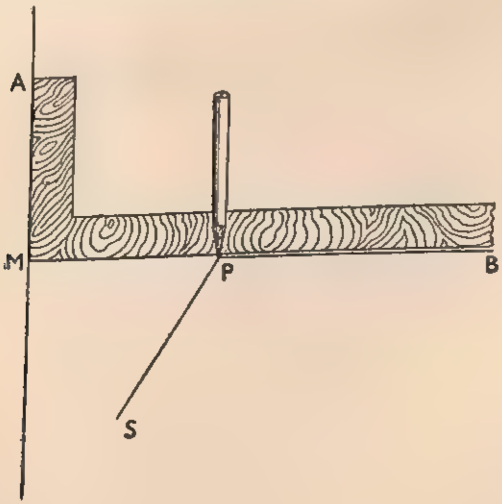
$$SP_1 = ZN_1 \textit{ (by construction)} = P_1M_1.$$

Similarly, for Q_1.

Taking other points N_2, N_3, etc., we similarly obtain the points P_2, Q_2, P_3, Q_3, etc.

The curve drawn through A, P_1, Q_1, P_2, Q_2, etc. is a parabola.

VII-3. Mechanical construction :



The edge of the shorter arm MA of an L-shaped rigid framework is

placed coincident with the directrix and one extremity of a string of length equal to the longer arm MB is tied at the end B of this arm, the other extremity being fastened at the focus S. The string is kept stretched by means of the point of a pencil at P in contact with the arm MB as in the figure. If the frame be now moved so that the arm MA slides along the directrix, the point of the pencil as it moves will describe a curve which is a parabola, for

$$SP+PB = \text{length of the string} = MB$$
$$= MP+PB$$

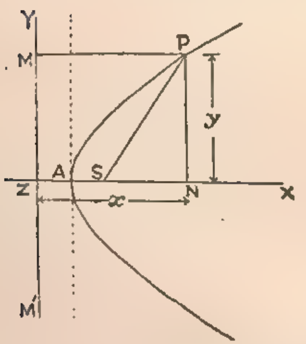
giving $$SP = MP.$$

Hence, P describes a parabola.

VII-4. Equation of the Parabola :

To find the equation to a parabola referred to the **axis and directrix as axes of coordinates.**

Let S be the focus and MM' the directrix. Draw SZ perpendicular to MM' and produce ZS to X. Then ZX is the axis of the parabola. Let ZX, ZM be taken as the axes of coordinates.



Suppose that the length $ZS = 2a$, so that the point S is $(2a, 0)$.

Let $P(x, y)$ be any point on the curve. Join SP and draw PN, PM perpendiculars to the axes. Then the condition satisfied by P is

$$SP = PM$$

$$\therefore \quad SP^2 = PM^2 = ZN^2$$

i.e., $$(x-2a)^2 + y^2 = x^2$$

or, $$\mathbf{y^2 = 4a(x-a)}, \text{ on reduction.}$$

This being the relation between the coordinates of any point on the curve is the required equation to the curve.

VII-5. The standard equation :

In the figure of the previous article, let A be the middle point of ZS, so that its coordinates are $(a, 0)$.

If we now transfer the origin to this point without changing

the directions of the axes, the equation is transformed into

$$y^2 = 4a(x + a - a)$$

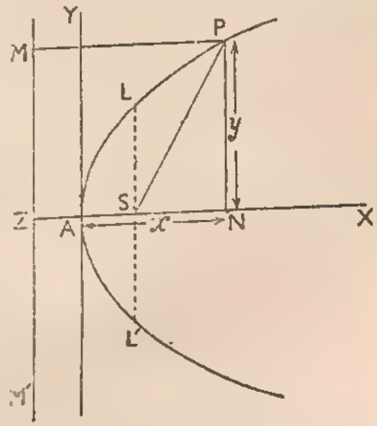
i.e., $$\mathbf{y^2 = 4ax}$$

which is the equation to the curve referred to AX and the line through A parallel to the directrix as axes of coordinates.

The choice of axes as has been made here leads to the simplest form of the equation to a parabola. It is therefore taken as the **Standard equation** to the curve.

We may, however, derive the standard equation independently thus :

Let S be the focus and MM' the directrix. Draw SZ perpendicular to MM' and bisect SZ at A.



Produce ZS to X. Draw AY perpendicular to AX.

We choose AX and AY as axes of coordinates.

Let $P(x, y)$ be any point on the curve.

Join SP and draw PM perpendicular to MM' and PN perpendicular to AX.

Let SZ be called $2a$, so that $SA = AZ = a$.

Now, the condition satisfied by P in order that it may be a point on the locus, is

$$SP = PM$$

$$\therefore \quad SP^2 = PM^2 = ZN^2,$$

i.e., $$(x-a)^2 + y^2 = (x+a)^2$$

which on reduction gives

$$\mathbf{y^2 = 4ax}$$

as the standard equation to the parabola.

VII-6. Definitions :

Axis.—The straight line (AX) drawn through the focus

perpendicular to the directrix is called the axis of the parabola.

Vertex.—The point (A) where the axis of the parabola meets the curve is called the vertex of the parabola.

Double ordinate.—A straight line drawn perpendicular to the axis of the parabola and terminated at both ends by the curve is called a double ordinate.

Latus rectum.—The double ordinate (LSL') passing through the focus of the parabola is called the latus rectum or the **principal parameter** of the curve.

It will be shown later that

$$LSL' = 4AS$$ [*Ref. Art. VII*-10]

VII-7. Property of the curve expressed by the standard equation :

The standard equation of the parabola corresponds to the geometrical property of the curve *viz.*, $PN^2 = 4AS.AN$, which may be stated as :

The square on the ordinate of any point on a parabola is equal to the rectangle contained by the latus rectum and the abscissa of the point.

VII-8. Shape of the curve :

Consider the standard equation $y^2 = 4ax$, where a is supposed positive.

(*i*) Corresponding to any positive value of x we get two equal and opposite values of y, showing that chords perpendicular to the axis of the curve are bisected by the axis and hence the curve is symmetrical about the axis of x.

(*ii*) When $x=0$, we get two equal values of y namely zero, showing that the line $x=0$ *i.e.*, the y-axis passes through two coincident points at A and hence is a tangent to the curve at the vertex.

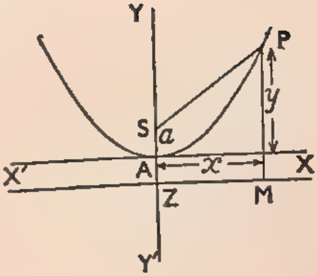
(*iii*) If x is negative y is imaginary ; hence there is no point of the curve to the left of the y-axis, the curve lying wholly to its right.

(*iv*) With the increase of x in magnitude y also increases, showing that the curve gradually recedes further and further away from the x-axis on both sides of it.

From the peculiarities of the curve obtained above we can form an idea as to its shape which is as shown in the figure of Art. VII-2.

VII-9. Other forms of the equation :

(1) If the axis of the curve is taken as the y-axis and the tangent at the vertex as the x-axis, then the relation $SP^2 = PM^2$ now gives



$$(y-a)^2 + x^2 = (y+a)^2,$$

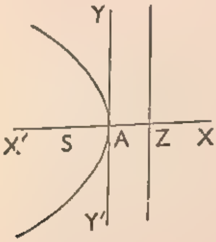
i.e., $$\mathbf{x^2 = 4ay}$$

as the required equation.

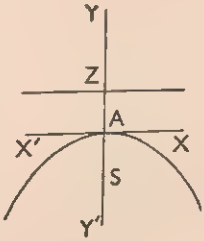
(2) If the concavity of the curve is towards the negative direction of the axis of x as in fig. (i), then since in this case the focus is to the left of the vertex and has coordinates $(-a, 0)$, a being positive, the equation to the parabola will be

$$\mathbf{y^2 = -4ax}.$$

Here, x is negative and the curve is entirely to the left of the y-axis.



(i)



(ii)

(3) Similarly, the equation is

$$\mathbf{x^2 = -4ay}$$

if the curve is concave towards the negative direction of the axis of y. [*Fig.* (ii)]

VII-10. Latus rectum (LSL′) :

From the equation $y^2 = 4ax$, since L is a point on the parabola, we have

$$SL^2 = 4a.AS$$
$$= 4a.a = 4a^2.$$

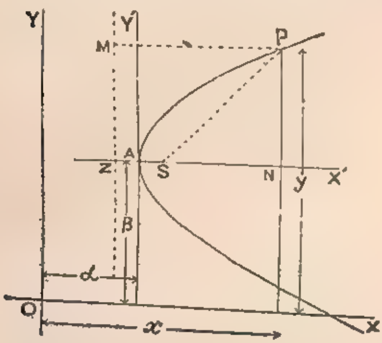
$\therefore$ $$SL = 2a.$$

Hence, $$\mathbf{LSL' = 4a}.$$

Cor. When the latus rectum of a parabola is given, the equation to the parabola is at once known in its standard form.

VII-11. Axis parallel to x or y-axis :

Let the coordinates of the vertex A be (α, β) and let the latus rectum be $4a$. Draw AX', AY' parallel to the axes.



(*i*) Let the axis of the parabola be parallel to the axis of x, *i.e.*, along AX'.

Clearly, with reference to AX' and AY' as axes of coordinates the equation to the parabola is

$$y^2 = 4ax$$

If now we transfer the origin from A to O retaining the directions of the axes, we get its equation referred to OX and OY as axes of coordinates.

The point A being (α, β), the coordinates of O referred to AX' and AY' as axes are clearly $(-\alpha, -\beta)$.

Hence the transformed equation is

$$\mathbf{(y-\beta)^2 = 4a(x-\alpha)}$$

This, therefore, is the required equation to the parabola whose vertex is (α, β) and whose axis is parallel to the axis of x.

(*ii*) If the axis of the parabola be parallel to the axis of y, then arguing as in (*i*), we write $x-\alpha$ and $y-\beta$ for x and y in the equation $x^2 = 4ay$ and obtain

$$\mathbf{(x-\alpha)^2 = 4a(y-\beta)}$$

as the required equation.

Otherwise : The equation of (*i*) can also be derived directly thus :

The point S is $(\alpha+a, \beta)$.

$\therefore \quad SP^2 = \{x-(\alpha+a)\}^2 + (y-\beta)^2.$

Also $\quad PM = ZN = ZA + AN = a + (x-\alpha).$

Hence, from the relation $SP^2 = PM^2$, we get

$$(x-\alpha-a)^2 + (y-\beta)^2 = (x-\alpha+a)^2.$$

$\therefore \qquad (y-\beta)^2=(x-\alpha+a)^2-(x-\alpha-a)^2,$

i.e., $\qquad (y-\beta)^2=4a(x-\alpha).$

VII-12. Form of the equation when the axis is parallel to x or y-axis :

From the equation $(y-\beta)^2=4a(x-\alpha)$, we have

$$y^2-2y\beta+\beta^2=4ax-4a\alpha,$$

or, $$4ax=y^2-2\beta y+\beta^2+4a\alpha,$$

i.e., $$x=\frac{1}{4a}y^2-\frac{\beta}{2a}y+\frac{\beta^2+4a\alpha}{4a}$$

which is of the form [axis ∥ x-axis]

$$\mathbf{x=Ay^2+By+C.}$$

Similarly, the equation $(x-\alpha)^2=4a(y-\beta)$ can be put in the form [axis ∥ y-axis]

$$\mathbf{y=Ax^2+Bx+C}$$

where A, B and C are constants.

VII-13. Determination of the vertex of

(*i*) $x=Ay^2+By+C$

(*ii*) $y=Ax^2+Bx+C.$

(*i*) The equation can be written as

$$y^2+\frac{B}{A}y=\frac{x}{A}-\frac{C}{A},$$

or, $$y^2+2.\frac{B}{2A}y+\frac{B^2}{4A^2}=\frac{x}{A}+\frac{B^2}{4A^2}-\frac{C}{A},$$

or, $$\left(y+\frac{B}{2A}\right)^2=\frac{x}{A}+\frac{B^2-4AC}{4A^2},$$

i.e., $$\left(y+\frac{B}{2A}\right)^2=\frac{1}{A}\left(x+\frac{B^2-4AC}{4A}\right).$$

Comparing this with the equation

$$(y-\beta)^2=4a(x-\alpha)$$

of which the vertex is (α, β), we at once obtain the coordinates of the vertex to be

$$-\frac{B^2-4AC}{4A} \text{ and } -\frac{B}{2A}.$$

(*ii*) As in (*i*) the equation can be written as

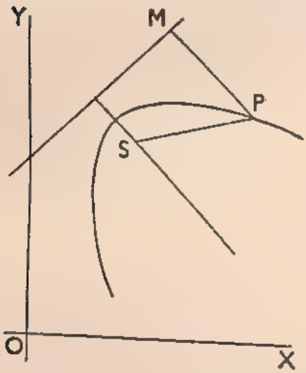
$$\left(x+\frac{B}{2A}\right)^2=\frac{1}{A}\left(y+\frac{B^2-4AC}{4A}\right)$$

whence, the coordinates of the vertex are

$$-\frac{B}{2A} \text{ and } -\frac{B^2-4AC}{4A}.$$

VII-14. General equation of the parabola :

To find the equation of the parabola when the coordinates of the focus and the equation to the directrix are given with reference to a set of rectangular axes.



Let S (α, β) be the focus and $Ax+By+C=0$ be the equation to the directrix.

If P (X, Y) be any point on the curve, then the geometrical condition satisfied by P is, $SP=PM$ where PM is the perpendicular from P to the directrix.

Now $\quad SP^2=(X-\alpha)^2+(Y-\beta)^2$

and $PM^2=\dfrac{(AX+BY+C)^2}{A^2+B^2}$.

$\therefore$ We have $(X-\alpha)^2+(Y-\beta)^2=\dfrac{(AX+BY+C)^2}{A^2+B^2}$.

Hence, (X, Y) satisfies the equation

$$(x-\alpha)^2+(y-\beta)^2=\frac{(Ax+By+C)^2}{A^2+B^2}$$

which is therefore the required equation to the parabola. The equation is of the second degree in x and y.

The above equation can be written as

$$(A^2+B^2)\{(x^2+y^2)-2(\alpha x+\beta y)+(\alpha^2+\beta^2)\}$$
$$-(Ax+By)^2-2C(Ax+By)-C^2=0.$$

If from the left-hand member of the equation we collect the terms of the second degree in x and y, the expression containing these terms

$$=(A^2+B^2)(x^2+y^2)-(Ax+By)^2$$
$$=A^2y^2+B^2x^2-2ABxy$$
$$=(Bx-Ay)^2$$

which is seen to be a perfect square.

This will be found to be the case in the equation to a

parabola in any of its forms. We therefore conclude that—

If the general equation of the second degree viz.,

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a parabola, the terms of the second degree in the equation

i.e., $ax^2+2hxy+by^2$

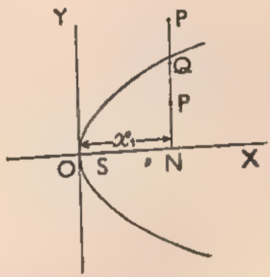
must form a perfect square, for which we must have

$$\mathbf{ab=h^2}$$

The general equation of a parabola is therefore of the form

$$\mathbf{(lx+my)^2+2gx+2fy+c=0.}$$

VII-15. Position of a point (x_1, y_1) with respect to the parabola $y^2=4ax$:



Let P be the point (x_1, y_1) and let PN drawn perpendicular to the axis meet the curve in Q.

Now Q being a point on the parabola whose abscissa is x_1, we have

$$QN^2=4ax_1$$

Clearly, the point $P(x_1, y_1)$ is outside, on, or inside the parabola according as $PN > =\text{or} < QN$

i.e., according as $PN^2 > =\text{or} < QN^2$

i.e., according as $y_1{}^2 > =\text{or} < 4ax_1$

i.e., according as the expression $y_1{}^2-4ax_1$ is positive, zero or negative.

Note : The curve divides the plane into two parts, one containing the focus and the other not containing it. We describe the former as 'inside the parabola' and the latter 'outside the parabola'.

WORKED OUT EXAMPLES

Ex. 1. *Find the vertex of the parabola*

$$3y^2+6y+4x+5=0.$$

Dividing both sides of the equation by 3, we have,

$$y^2+2y=-\tfrac{4}{3}x-\tfrac{5}{3},$$

or, $\qquad (y+1)^2 = -\frac{4}{3}x - \frac{5}{3} + 1$

$$= -\frac{4}{3}x - \frac{2}{3},$$

or, $\qquad (y+1)^2 = -\frac{4}{3}(x+\frac{1}{2}).$

Comparing this with the equation $(y-\beta)^2 = 4a(x-\alpha)$ of which the vertex is (α, β), we get the required vertex to be the point $(-\frac{1}{2}, -1)$.

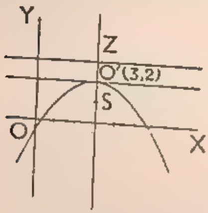
Ex. 2. *Find the latus rectum, vertex, focus, axis and directrix of the parabola*

$$x^2 - 6x + 4y + 1 = 0.$$

The equation can be written as

$$(x-3)^2 = -4(y-2)$$

which shows that the vertex is the point (3, 2).



Referred to parallel axes through $O'(3, 2)$ the equation reduces to

$$x^2 = -4y$$

Comparing this with the equation

$$x^2 = -4ay \qquad [\text{ Art. VII-9 (3) }]$$

we find that the curve is concave downwards, the latus rectum $(=4a)$ being of length 4 and the distance between the vertex and focus $(=a)$ being equal to 1.

If then S be the focus, $O'S = 1$, so that the coordinates of the focus are 3 and 1.

The axis is a line through $O'(3, 2)$ parallel to the y-axis and hence its equation is $x=3$ while the directrix is a line through Z parallel to the x-axis where $O'Z = O'S = 1$, its equation being $y = 3$.

We therefore get the following results :

(*i*) The latus rectum $=4$.

(*ii*) The vertex is the point (3, 2).

(*iii*) The focus is the point (3, 1).

(*iv*) The axis is the line $x=3$.

(*v*) The directrix is the line $y=3$.

Ex. 3. *Find the equation of the parabola whose focus is the point (2, 1) and whose directrix is the straight line* $4x-3y=1$, *and determine the length of its latus rectum.*

If (x, y) be any point on the curve then the distance of (x, y) from the point $(2, 1)$ must be equal to its distance from the line $4x-3y=1$. This gives

$$\sqrt{(x-2)^2+(y-1)^2}=\frac{4x-3y-1}{\sqrt{4^2+3^2}}.$$

Squaring both sides, we get

$$x^2+y^2-4x-2y+5=\frac{16x^2+9y^2-24xy-8x+6y+1}{25}$$

which reduces to

$$9x^2+24xy+16y^2-92x-56y+124=0.$$

This is the required equation.

Also the length of the perpendicular from the focus on the directrix is

$$\frac{4.2-3.1-1}{\sqrt{4^2+3^2}}=\frac{4}{5}.$$

Hence, the latus rectum which is twice this distance is $\frac{8}{5}$.

EXERCISE VII(A)

1. Find the equations of the following parabolas :
   (*i*) Focus at $(3, 0)$, directrix $x=-3$;
   (*ii*) Focus at $(0, 0)$, directrix $x=-6$;
   (*iii*) Focus at $(-6, 0)$, directrix $x=0$;
   (*iv*) Focus at $(0, 6)$, directrix $y=0$;
   (*v*) Focus at $(0, -3)$, directrix $y=3$;
   (*vi*) Focus at $(0, 0)$, directrix $y=6$.
2. Find the equations of the following parabolas :
   (*i*) Focus at $(-1, 2)$, directrix $x=-5$;
   (*ii*) Focus at $(-1, -2)$, directrix $x=3$;
   (*iii*) Focus at $(1, 1)$, directrix $y=-3$;
   (*iv*) Focus at $(-1, -2)$, directrix $y=2$.
3. Find the equation to the parabola :
   (*i*) Whose focus is at the origin and whose directrix is the straight line $2x+y-1=0$. [*C. U.* 1954]
   (*ii*) Whose focus is the point $(-1, 1)$ and whose directrix is the straight line $x+y+1=0$. [*C. U.*]

   Also determine the length of the latus rectum.

4. The parabola $y^2=4px$ goes through the point $(3, -2)$. Obtain the length of the latus rectum and the coordinates of the focus. [C. U.]

5. Find the vertex of the parabola :

(i) $x^2-6x-8y-7=0$;

(ii) $2y^2-6y+5x+7=0$.

6. Prove that the equation

$$y^2+2ax+2by+c=0$$

represents a parabola whose axis is parallel to the axis of x. Find its vertex. [C. U. 1953]

7. In the parabola

$$4(y-1)^2=-7(x-3)$$

find (i) the latus rectum and (ii) the coordinates of the focus and vertex. [C. U. 1956]

8. Find the latus rectum, vertex, focus, axis and directrix of the following parabolas :

(i) $y^2-2y+8x-23=0$;

(ii) $x^2+8x+12y+4=0$.

9. Find the equation to the parabola whose axis is parallel to the y-axis and which passes through the points $(0, 4)$, $(1, 9)$ and $(-2, 6)$ and determine its latus rectum.

10. Show that the latus rectum of a parabola is a third proportional to any abscissa and its corresponding ordinate. [C. U.]

11. A double ordinate of the curve $y^2=4px$ is of length $8p$; prove that the lines from the vertex to its two ends are at right angles. [C. U. 1952]

Answers

1. (i) $y^2=12x$; (ii) $y^2=12(x+3)$; (iii) $y^2=-12(x+3)$; (iv) $x^2=12(y-3)$; (v) $x^2=-12y$; (vi) $x^2=-12(y-3)$.

2. (i) $(y-2)^2=8(x+3)$; (ii) $(y+2)^2=-8(x-1)$; (iii) $(x-1)^2=8(y+1)$; (iv) $(x+1)^2=-8y$.

3. (i) $x^2+4y^2-4xy+4x+2y-1=0, \frac{2}{5}\sqrt{5}$;
(ii) $x^2+y^2-2xy+2x-6y+3=0, \sqrt{2}$.

4. $\frac{4}{3}, (\frac{1}{3}, 0)$. 5. (i) $(3, -2)$; (ii) $(-\frac{1}{2}, \frac{3}{2})$.

6. $\left(\frac{b^2-c}{2a}, -b\right)$. 7. (i) $\frac{7}{4}$; (ii) focus $(\frac{41}{16}, 1)$, vertex $(3, 1)$.

8. (i) Latus rectum $=8$, vertex $(3, 1)$, focus $(1, 1)$, axis $y=1$, directrix $x=5$.

(ii) Latus rectum $=12$, vertex $(-4, 1)$, focus $(-4, -2)$, axis $x+4=0$, directrix $y=4$.

9. $y=2x^2+3x+4$; $\frac{1}{2}$.

VII-16. Tangent at a point :

To find the equation of the tangent to the parabola $y^2=4ax$ at the point (x_1, y_1).

Let P be the given point (x_1, y_1). We take a point Q on the parabola close to P. Let its coordinates be (x_2, y_2).

The equation to PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \cdots \quad \cdots \quad (1)$$

But $\quad y_1{}^2=4ax_1$ and $y_2{}^2=4ax_2$

whence $\quad y_2{}^2-y_1{}^2=4a(x_2-x_1)$

$$\therefore \quad \frac{y_2-y_1}{x_2-x_1}=\frac{4a}{y_2+y_1} \quad \cdots \quad \cdots \quad (2)$$

Substituting from (2) in (1) we get

$$y-y_1=\frac{4a}{y_2+y_1}(x-x_1)$$

which then is *the equation of the chord PQ* of the parabola.

If now Q tends to coincidence with P, the chord in this limiting position becomes the tangent at P. The equation of the tangent, obtained by putting $y_2=y_1$ in the above equation is thus

$$y-y_1=\frac{2a}{y_1}(x-x_1)$$

i.e., $\quad yy_1-y_1{}^2=2ax-2ax_1$

i.e., $\quad \mathbf{yy_1=2a(x+x_1)}$ $\quad$ since $y_1{}^2=4ax_1$.

Alternative method :

If (x, y) be a point on the curve and $(x+\delta x, y+\delta y)$ another point on it close to (x_1, y_1), then

$$(y+\delta y)^2=4a(x+\delta x) \quad \text{and} \quad y^2=4ax$$

$\therefore$ By subtraction, $\quad 2y\delta y+\delta y^2=4a\delta x$

whence $\quad \dfrac{\delta y}{\delta x}=\dfrac{4a}{2y+\delta y}$

$$\therefore \quad \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta y\to 0}\frac{4a}{2y+\delta y}, \quad \because \delta y\to 0 \text{ when } \delta x\to 0, =\frac{2a}{y}$$

The gradient of the tangent at (x_1, y_1) is therefore $\dfrac{2a}{y_1}$.

The equation of the tangent is then

$$y-y_1=\frac{2a}{y_1}(x-x_1)$$

whence the equation

$$yy_1=2a(x+x_1)$$

readily follows.

Note : It will be seen that the rule to write down the equation of the tangent to the circle as laid down in Art. VI-11, Note, applies to the case of the tangent to the parabola as well.

VII-17. Condition for tangency :

To find the condition that the straight line $y=mx+c$ should touch the parabola $y^2=4ax$.

The points common to the line and the curve are found by solving the equations

$$y^2=4ax \text{ and } y=mx+c.$$

The abscissæ of the points of intersection are therefore given by the roots of

$$(mx+c)^2=4ax,$$

i.e., of $m^2x^2+2x(mc-2a)+c^2=0.$ ... ... (1)

If the line is a tangent, the two points of intersection coincide and hence the equation (1) must have equal roots.

The condition for this is

$$4(mc-2a)^2=4m^2c^2,$$

i.e., $-4amc+4a^2=0,$

or, $c=\dfrac{a}{m}.$

Remark 1. The equation (1) being quadratic in x has two roots and so the line meets the parabola in two points. These points of intersection are real or imaginary according as the roots of (1) are real or imaginary, *i.e.*, according as $4(mc-2a)^2$ $>$ or $< 4m^2c^2$, *i.e.*, according as $-4amc+4a^2 >$ or < 0, *i.e.*, according as $mc <$ or $> a$, *i.e.*, $c <$ or $> \dfrac{a}{m}$. When the roots are imaginary, geometrically the line does not meet the curve at all.

Remark 2. The equation (1) has one root infinite if the coefficient of x^2 is zero, *i.e.*, if $m=0$. But when $m=0$, the equation of the straight line reduces to $y=c$. Hence, we conclude that *a line parallel to the axis of the parabola meets it in one point at an infinite distance from the origin.*

VII-18. Tangent in a given direction :

If the gradient m of a line is given, we find from the previous article that the value $\frac{a}{m}$ of c will make $y=mx+c$ a tangent to the parabola $y^2=4ax$. Hence, corresponding to any given gradient m (except when $m=0$), there is always a tangent to the parabola given by the equation

$$\mathbf{y=mx+\frac{a}{m}}.$$

VII-19. Point of contact :

The line $y=mx+\frac{a}{m}$ ... (1) is always a tangent to the parabola $y^2=4ax$ ... (2).

If we solve the equations (1) and (2) we shall get the two coincident points in which the line (1) meets the curve (2) *i.e.*, the point of contact. Substituting for y, we get

$$\left(mx+\frac{a}{m}\right)^2-4ax=0,$$

or,
$$\left(mx-\frac{a}{m}\right)^2=0.$$

$\therefore$ $x=\frac{a}{m^2}$ which is the abscissa of the point of contact.

Substituting for x in (1), we have $y=\frac{2a}{m}$ giving the ordinate.

The point of contact is therefore

$$\left(\frac{\mathbf{a}}{\mathbf{m}^2}, \frac{\mathbf{2a}}{\mathbf{m}}\right).$$

VII-20. Number of tangents from a point :

To prove that two tangents, real or imaginary can be drawn from a point to a parabola.

Let the equation to the parabola be $y^2=4ax$ and let (x_1, y_1) be the given point.

The line $$y=mx+\frac{a}{m} \qquad \dots \qquad \dots \quad (1)$$

is always a tangent to the parabola $y^2=4ax$, and for different values of m it represents tangents in different directions. We are required to prove that two of these pass through the given point (x_1, y_1).

If (1) passes through (x_1, y_1), we have

$$y_1=mx_1+\frac{a}{m},$$

or, $$m^2x_1-my_1+a=0. \qquad \dots \quad (2)$$

The above equation is quadratic in m and therefore gives two values of m corresponding to each of which we have a tangent passing through (x_1, y_1).

Hence, two tangents can be drawn from the given point to the parabola and these are

$$\mathbf{y=m_1x+\frac{a}{m_1}}$$

and $$\mathbf{y=m_2x+\frac{a}{m_2}}$$

where m_1 and m_2 are the roots of the equation (2).

From equation (2),

(*i*) If $y_1^2-4ax_1$ is positive, *i.e.*, the point (x_1, y_1) is outside the parabola [*Art. VII*-15], we get two real and distinct roots and hence two distinct tangents can be drawn.

(*ii*) If $y_1^2-4ax_1$ is zero, *i.e.*, the point is on the parabola, we get two equal roots and hence the two tangents become coincident.

(*iii*) If $y_1^2-4ax_1$ is negative, *i.e.*, the point (x_1, y_1) is inside the parabola, we get two imaginary roots and in this case both the tangents are imaginary *i.e.*, geometrically no tangents can be drawn.

VII-21. Normal at a point :

To find the equation of the normal to the parabola $y^2=4ax$ at the point (x_1, y_1).

The equation of the tangent at (x_1, y_1) is

$$yy_1=2a(x+x_1),$$

i.e.,
$$y=\frac{2a}{y_1}x+\frac{2ax_1}{y_1}. \quad \dots \quad \dots \quad (1)$$

$\therefore$ The equation of the normal which is a straight line through (x_1, y_1) perpendicular to the tangent (1) is

$$y-y_1=m(x-x_1) \quad \dots \quad \dots \quad (2)$$

where
$$m\times\frac{2a}{y_1}=-1, \qquad i.e., \qquad m=-\frac{y_1}{2a}.$$

Hence, substituting for m in (2) the required equation is

$$\mathbf{y-y_1=-\frac{y_1}{2a}(x-x_1).} \quad \dots \quad \dots \quad (3)$$

VII-22. Equation of the normal in terms of its gradient :

In equation (3) of the last article, we put

$$-\frac{y_1}{2a}=m \quad \dots \quad \dots \quad \dots \quad (1)$$

m therefore represents the gradient of the normal.

From (1), $y_1=-2am$.

Since, (x_1, y_1) is a point on the parabola,

$$x_1=\frac{y_1^2}{4a}=am^2.$$

The equation of the normal then takes the form

$$y+2am=m(x-am^2),$$

i.e.,
$$\mathbf{y=mx-2am-am^3.} \quad \dots \quad \dots \quad (2)$$

Also the foot of the normal, that is, the point of the curve at which (2) is a normal has coordinates

$$\mathbf{(am^2, -2am)}$$

VII-23. Number of normals from a point :

To prove that, in general, three normals can be drawn from a point to a parabola.

Let (x_1, y_1) be the given point.

The equation $y = mx - 2am - am^3$ ... ... (1)

is a normal to the parabola $y^2 = 4ax$.

We have now to choose m so that (1) may pass through the point (x_1, y_1).

We get $y_1 = mx_1 - 2am - am^3$,

or, $am^3 + (2a - x_1)m + y_1 = 0$. ... ... (2)

The equation being of the third degree in m gives three values (real or imaginary) of m corresponding to each of which we have a normal passing through the point (x_1, y_1).

Hence, **three** normals can be drawn from a point to a parabola and these are

$$y = m_1x - 2am_1 - am_1^3$$
$$y = m_2x - 2am_2 - am_2^3$$
$$y = m_3x - 2am_3 - am_3^3$$

where m_1, m_2 and m_3 are the three roots of the equation (2).

The feet of these three normals are respectively

$$(am_1^2, -2am_1), (am_2^2, -2am_2), (am_3^2, -2am_3).$$

VII-24. Co-normal points :

Def. : The feet of the three normals drawn from a point to a parabola, that is, the points of the curve the normals at which meet in a point are called co-normal points.

VII-25. An important property of co-normal points :

From the equation (2) of *Art. VII*-23, we get (since there is no term containing m^2)

$$m_1 + m_2 + m_3 = 0.$$

$$\therefore \quad -2am_1 - 2am_2 - 2am_3 = 0,$$

i.e., the algebraic sum of the ordinates of the three co-normal points is zero.

WORKED OUT EXAMPLES

Ex. 1. *Find the length of the chord intercepted by the parabola $y^2=4ax$ on the straight line $y=mx+c$.*

The points common to the straight line and the parabola are found by solving the two equations. If then (x_1, y_1) and (x_2, y_2) are the common points, x_1 and x_2 must be the roots of the equation

$$(mx+c)^2=4ax,$$

i.e., $$m^2x^2+2(mc-2a)x+c^2=0.$$

Hence, $$(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$

$$=\frac{4(mc-2a)^2}{m^4}-\frac{4c^2}{m^2}$$

$$=\frac{4}{m^4}\left\{(mc-2a)^2-m^2c^2\right\}$$

$$=\frac{16a(a-mc)}{m^4}.$$

$\therefore$ $$x_1-x_2=\frac{4}{m^2}\sqrt{a(a-mc)}.$$

Again, since (x_1, y_1) and (x_2, y_2) are points on the line $y=mx+c$, we have

$$y_1=mx_1+c \text{ and } y_2=mx_2+c.$$

$\therefore$ $$y_1-y_2=m(x_1-x_2).$$

Hence, the required chord

$$=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$$=(x_1-x_2)\sqrt{1+m^2}$$

$$=\frac{4}{m^2}\sqrt{a(a-mc)(1+m^2)}.$$

Ex. 2. *Find the equations of the tangent and normal to the parabola $y^2=12x$ at the point $(3, -6)$.*

Comparing the equation with $y^2=4ax$, we get $a=3$.

Hence, on substitution in $yy_1=2a(x+x_1)$, we get

$$y(-6)=2.3\,(x+3),$$

i.e., $$x+y+3=0$$

as the required equation to the tangent.

The equation to the normal is

$$y-(-6)=\frac{-(-6)}{2.3}(x-3),$$

i.e., $\quad y+6=x-3,$

i.e., $\quad x-y=9.$

Ex. 3. *Prove that the straight line $x+y=1$ touches the parabola $y^2-y+x=0$, and find the point of contact.*

The ordinates of the points in which the line meets the parabola are given by the roots of the equation

$$y^2-y+(1-y)=0,$$

i.e., $\quad (y-1)^2=0.$

$\therefore \quad y=1, 1.$

The roots being equal, the two points of intersection coincide and hence, the line is a tangent.

Putting $y=1$ in the equation $x+y=1$, we get $x=0$.

Hence, the point of contact is $(0, 1)$.

Ex. 4. *Find the point of the parabola $y^2=4ax$, at which the normal is inclined at 30° to the axis.* [*C. U.*]

We have, $\quad m=$ gradient of the normal

$=\tan 30^\circ$

$=\dfrac{1}{\sqrt{3}}.$

Hence, the required point, *i.e.*, the point at which the normal has a gradient m has coordinates am^2 and $-2am$

[*Art. VII-22*]

i.e., $\quad \dfrac{a}{3}$ and $-\dfrac{2a}{\sqrt{3}}.$

Ex. 5. *Find the equation of the common tangent to the two parabolas $y^2=4ax$ and $x^2=4by$.*

The equation of any tangent to the parabola $y^2=4ax$ is

$$y=mx+\frac{a}{m}.$$

This line will touch the other parabola $x^2 = 4by$, if the roots of

$$x^2 = 4b\left(mx + \frac{a}{m}\right),$$

i.e., of $mx^2 - 4bm^2x - 4ab = 0$ be equal.

The condition for this is

$$16b^2m^4 = 4.m.(-4ab),$$

i.e., $$m^3 = -\frac{a}{b},$$

i.e., $$m = -\left(\frac{a}{b}\right)^{\frac{1}{3}}.$$

Hence, the required equation of the common tangent is,

$$y = -\left(\frac{a}{b}\right)^{\frac{1}{3}}x - a\left(\frac{b}{a}\right)^{\frac{1}{3}},$$

i.e., $$q^{\frac{1}{3}}y = -a^{\frac{1}{3}}x - a^{\frac{2}{3}}b^{\frac{2}{3}},$$

i.e., $$a^{\frac{1}{3}}x + b^{\frac{1}{3}}y + a^{\frac{2}{3}}b^{\frac{2}{3}} = 0.$$

EXERCISE VII(B)

1. Find the equation of the tangent and normal to
   (*i*) $y^2 = 4x$ at $(1, 2)$;
   (*ii*) $x^2 = -12y$ at $(6, -3)$;
   (*iii*) $y^2 = 8x$ at the ends of the latus rectum ;
   (*iv*) $x^2 + 2x + y = 4$ at $(-2, 4)$.

2. Two equal parabolas have the same vertex and their axes are at right angles. Prove that they cut again at an angle $\tan^{-1}\frac{3}{4}$. [*C. U.*]

3. Prove that $y = \frac{1}{2}x + 2a$ is a tangent to the parabola $y^2 = 4ax$.

4. Obtain the equation of the tangent to the parabola $y^2 = 4ax$ which makes an angle 60° with the axis of x.

5. Find the equation of the tangent to the parabola $3y^2 = 4x$ which is perpendicular to the straight line $3x + 4y + 5 = 0$; also determine the point of contact.

6. A tangent to the parabola $y^2 = 8x$ makes an angle 45° with the straight line $y = 3x + 5$. Find its equation and its points of contact. [*C. U.*]

7. Find the coordinates of the point of intersection of the two tangents $y = mx + \dfrac{a}{m}$ and $y = m_1x + \dfrac{a}{m_1}$ to the parabola $y^2 = 4ax$ and deduce the condition that this point may lie on the latus rectum. [*C. U.*]

8. Find the condition that the straight line $lx+my+n=0$ may touch the parabola $y^2=4ax$. [*C. U.*]

9. Show that the line $y=mx+c$ touches the parabola $y^2=4a(x+a)$ if $c=am+\dfrac{a}{m}$ and hence, find the tangent to the parabola $y^2=8(x+2)$ which makes an angle 45° with the axis of x.

10. Two equal parabolas have the same vertex and their axes are at right angles; prove that the common tangent touches each at the end of a latus rectum. [*C. U.*]

11. Find the equation of the common tangent to the two parabolas $y^2=32x$ and $x^2=108y$. [*C. U.*]

12. A straight line touches both $x^2+y^2=2a^2$ and $y^2=8ax$; show that its equation is $y=\pm(x+2a)$. [*C. U.* 1955]

13. For the parabola $y^2=8x$ form the equation of the two tangents which pass through the point $(-2, \frac{16}{3})$. Also find the angle included between them. [*C. U.* 1957]

14. Prove that the normal to the circle $x^2+y^2+4x+2y-8=0$ at the point (1, 1) is a tangent to the parabola $9y^2=8x$.

15. Obtain the equation of the tangent and normal at the point $(am^2, 2am)$ on the parabola $y^2=4ax$.

The normal to the parabola at $(am_1{}^2, 2am_1)$ meets the curve again at $(am_2{}^2, 2am_2)$; prove that $m_1{}^2+m_1m_2+2=0$. [*C. U.*]

16. Find the point of the parabola $y^2=8x$ at which the normal is parallel to the straight line $x-2y+3=0$; also determine the equation to the normal.

17. Find the coordinates of that particular point on the parabola $y^2=4ax$, the normal at which,—terminated by the axis, is equal in length to the latus rectum. [*C. U.* 1956]

18. Find the length of the normal chord of the parabola $y^2=4x$ at the point whose ordinate is equal to its abscissa.

19. If $(at_1{}^2, 2at_1)$ and $(at_2{}^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2=4ax$, show that $t_1\,t_2=-1$.

Answers :

1. (*i*) $y=x+1$, $x+y=3$; (*ii*) $x+y=3$, $x-y=9$; (*iii*) $x-y+2=0$, $x+y=6$ and $x+y+2=0$, $x-y=6$; (*iv*) $y=2x+8$, $x+2y=6$.

4. $3x-\sqrt{3}y+a=0$. 5. $16x-12y+3=0$, $(\frac{3}{16}, \frac{1}{2})$.

6. $x-2y+8=0$ and $2x+y+1=0$; points of contact (8, 8) and $(\frac{1}{2}, -2)$ respectively.

7. $\left[\dfrac{a}{mm'}, a\left(\dfrac{1}{m}+\dfrac{1}{m'}\right)\right]$, $mm'=1$. 8. $ln=am^2$. 9. $y=x+4$.

11. $2x+3y+36=0$. 13. $x-3y+18=0$, $9x+3y+2=0$; 90°.

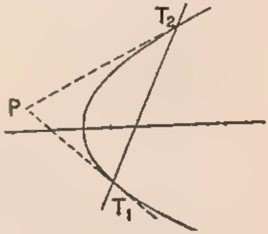
15. $my=x+am^2$, $y+mx=2am+am^3$. 16. $(\frac{1}{2}, -2)$, $2x-4y=9$.

17. $(3a, \pm 2\sqrt{3}a)$. 18. $5\sqrt{5}$.

VII-26. Chord of contact :

To find the equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$.

Let $T_1\ (\alpha_1, \beta_1)$ and $T_2(\alpha_2, \beta_2)$ be the points of contact of tangents from the given point $P\,(x_1, y_1)$ to the given parabola.



The tangents at T_1 and T_2 are respectively

$$y\beta_1 = 2a(x+\alpha_1)$$

and $$y\beta_2 = 2a(x+\alpha_2)$$

Since these tangents pass through $P\,(x_1, y_1)$, we have $y_1\beta_1 = 2a(x_1+\alpha_1)$

and $$y_1\beta_2 = 2a(x_1+\alpha_2).$$

We find that the coordinates (α_1, β_1) and (α_2, β_2) both satisfy the equation

$$yy_1 = 2a(x+x_1)$$

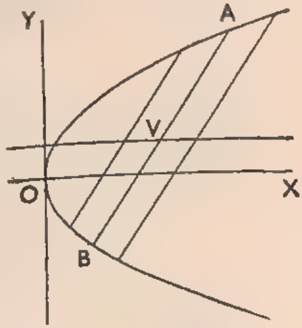
Hence, the points T_1 and T_2 lie on the straight line

$$\mathbf{yy_1 = 2a(x+x_1)}$$

which is therefore the required equation of the chord of contact.

VII-27. Diameter of the parabola :

To find the locus of the middle points of a system of parallel chords of the parabola $y^2 = 4ax$. ... (1)



If m be the gradient of the system of parallel chords, the equation of any one of the system, say AB, is given by

$$y = mx + c, \quad \cdots \quad (2)$$

different values of c giving different chords of the system.

If $A(x_1, y_1)$ and $B\,(x_2, y_2)$ be the points in which the line (2) meets the parabola (1), then y_1 and y_2 are the roots of

$$y^2 = 4a\left(\frac{y-c}{m}\right),$$

i.e., of $$my^2 - 4ay + 4ac = 0.$$

If (h, k) be the coordinates of V, the mid-point of AB, then

$$k=\frac{y_1+y_2}{2}=\frac{2a}{m}.$$

The result being independent of c must be true for all values of c, that is, for all the chords of the system. In other words, the ordinates of the mid-points of all the parallel chords of the system will be the same and equal to $\frac{2a}{m}$. Hence, the locus is

$$\mathbf{y}=\frac{\mathbf{2a}}{\mathbf{m}}$$

which is a straight line parallel to the axis of x.

Definition. *The locus of the middle points of a system of parallel chords of a parabola is called a* **diameter.**

VII-28. Chord having its middle point given :

To find the equation of the chord of the parabola $y^2=4ax$ which is bisected at a given point (α, β).

Since, the required chord passes through (α, β) its equation must be of the form

$$y-\beta=m(x-\alpha) \quad \cdots \quad \cdots \quad (1)$$

where m has to be found out.

Also (α, β) being the middle point of the chord must lie on the diameter corresponding to the system of chords parallel to (1).

The equation to this diameter is $y=\frac{2a}{m}$.

Hence, $\beta=\frac{2a}{m}$ from which, $m=\frac{2a}{\beta}$.

$\therefore$ Substituting for m in (1), the required equation is

$$\mathbf{y}-\beta=\frac{\mathbf{2a}}{\beta}(\mathbf{x}-\alpha).$$

VII-29. Some Geometrical properties :

1. *The tangent to a parabola at its point of intersection with a diameter is parallel to the system of chords bisected by the diameter.*

The diameter corresponding to the system of parallel chords $y=mx+c$ where m is constant and c is different for different chords is given by

$$y=\frac{2a}{m}.$$

It meets the parabola $y^2=4ax$ where

$$\frac{4a^2}{m^2}=4ax, \qquad i.e., \ x=\frac{a}{m^2}.$$

$\therefore$ The point of intersection is $\left(\frac{a}{m^2}, \ \frac{2a}{m}\right)$.

The tangent at this point is

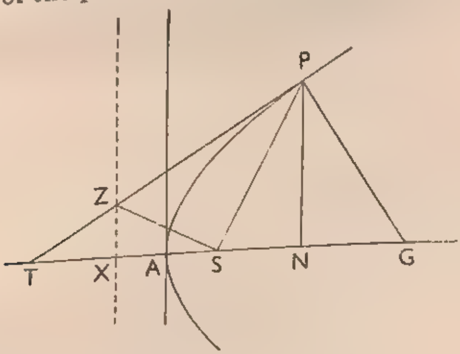
$$y.\frac{2a}{m}=2a\left(x+\frac{a}{m^2}\right),$$

i.e.,
$$y=mx+\frac{a}{m}$$

and is therefore parallel to the given system.

2. *The subtangent of any point on a parabola is bisected at the vertex.* $(AT=AN)$

Def. : The **subtangent** of any point on a parabola (or conic) is the portion of the axis intercepted between the tangent and the ordinate of the point.



Let $P(x_1, y_1)$ be a point on the parabola $y^2=4ax$.
The equation of PT the tangent at P is

$$yy_1=2a(x+x_1).$$

At T, where this tangent intersects the x-axis,

$y=0.$ $\therefore$ $x=-x_1$ (the negative sign indicating that T is to the left of A)

$\therefore$ $AT=x_1$ (*numerically*), also $AN=x_1$.

$\therefore$ $AT=AN$

which proves the proposition.

3. *The subnormal of any point on a parabola is constant and equal to the semi-latus rectum.* ($NG=2AS$)

Def. : The **subnormal** of any point on a parabola (or conic) is the portion of the axis intercepted between the normal and the ordinate of the point.

Let $P(x_1, y_1)$ be any point on the parabola $y^2=4ax$.

[See *fig.* of *Prop.* 2]

The equation of PG, the normal at P is

$$y-y_1=-\frac{y_1}{2a}(x-x_1).$$

At G, where this normal intersects the x-axis

$y=0.$ $\therefore$ $x=x_1+2a$

i.e., $AG=AN+2a.$

$\therefore$ $NG=2a$ (constant)

$=\frac{1}{2}\times 4a$

$=\frac{1}{2}\times$ latus rectum.

4. *If the tangent and normal at any point P of a parabola meet the axis in T and G respectively*, then

ST = SG = SP.

Let $P(x_1, y_1)$ be any point on the parabola $y^2=4ax$.

[See *fig. of Prop.* 2]

From Prop. 2, $AT=x_1$. $\therefore$ $ST=x_1+a.$

From Prop. 3, $AG=x_1+2a.$ $\therefore$ $SG=x_1+a.$

Also, the focus S being the point $(a, 0)$,

we have $SP=\sqrt{(x_1-a)^2+y_1^2}$

$=\sqrt{(x_1-a)^2+4ax_1}$ [$\because$ (x_1, y_1) is a point on the parabola]

$=x_1+a.$

Hence, $ST=SG=SP,$

5. *The portion of the tangent at any point of a parabola intercepted between the point of contact and the directrix subtends a right angle at the focus.* ($\angle PSZ = 90°$)

Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4ax$.

[See *fig. of Prop.* 2]

The equation of the tangent at P is

$$yy_1 = 2a(x + x_1) \qquad \ldots \qquad \ldots \quad (1)$$

The equation of the directrix is

$$x = -a \qquad \ldots \qquad \ldots \quad (2)$$

The point Z where these lines intersect is found by solving (1) and (2).

Substituting for x from (2) in (1),

$$yy_1 = 2a(-a + x_1),$$

i.e., $$y = \frac{2a}{y_1}(x_1 - a).$$

$\therefore$ The point Z has coordinates, $-a, \dfrac{2a}{y_1}(x_1 - a)$.

Gradient of SZ is $\dfrac{\frac{2a}{y_1}(x_1 - a) - 0}{-a - a}$ *i.e.,* $-\dfrac{x_1 - a}{y_1}$.

Gradient of SP is $\dfrac{y_1 - 0}{x_1 - a}$ *i.e.,* $\dfrac{y_1}{x_1 - a}$

Product of the gradients $= -\dfrac{x_1 - a}{y_1} \times \dfrac{y_1}{x_1 - a} = -1$.

Hence, SP and SZ are at right angles, which proves the proposition.

VII-30. Locus Problems :

Problem 1. *To find the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$, which meet at right angles.*

Let $P(h, k)$ be a point on the locus. Then the two tangents of the parabola which pass through P must be at right angles.

Now, $y = mx + \dfrac{a}{m}$ is always a tangent to the parabola $y^2 = 4ax$.

If it passes through $P(h, k)$, then

$$k = mh + \frac{a}{m}, \quad i.e., \quad m^2h - mk + a = 0 \qquad \dots \quad (1)$$

If m_1 and m_2 be the two roots of this equation, then the two tangents which pass through (h, k) are

$$y = m_1x + \frac{a}{m_1} \quad \text{and} \quad y = m_2x + \frac{a}{m_2}.$$

If these be at right angles, we have $m_1m_2 = -1$. Hence, from equation (1),

$$\frac{a}{h} = -1, \quad \text{or,} \quad h + a = 0.$$

The required locus is therefore the straight line

$$x + a = 0$$

which is the *directrix* of the parabola.

Problem 2. *To find the locus of the middle points of chords of the parabola $y^2 = 4ax$ which pass through a fixed point (α, β).*

If $P(h, k)$ be a point on the locus, then the chord having P its middle point must pass through (α, β).

Equation to the chord having (h, k) its middle point is

$$y - k = \frac{2a}{k}(x - h).$$

If it passes through (α, β), then $\beta - k = \dfrac{2a}{k}(\alpha - h)$,

i.e., $\beta k - k^2 = 2a\alpha - 2ah$,

i.e., $k^2 - \beta k = 2a(h - \alpha)$.

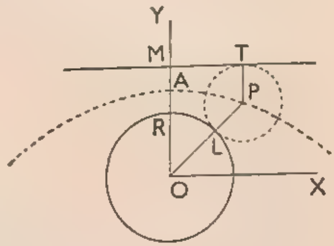
$\therefore$ The required locus of (h, k) is given by

$$y^2 - \beta y = 2a(x - \alpha)$$

which is clearly a parabola whose latus rectum is half the latus rectum of the given parabola.

Problem 3. *If a circle be drawn so as always to touch a given straight line and also a given circle, then the locus of its centre is a parabola.*

Let the centre of the circle be taken as the origin and lines through the centre parallel and perpendicular to the given line as the axes of x and y respectively.



Let the given line be at a distance d from the centre of the circle so that its equation is

$$y-d=0.$$

If $P\ (h,\ k)$ be the centre of the moving circle in any position and, L and T be respectively its points of contact with the given circle and the given straight line, then clearly

$$PO-OL=PT,$$

i.e., $\sqrt{h^2+k^2}-a=d-k$, where a is the radius of the given circle.

$\therefore\quad h^2+k^2=(d+a-k)^2,$

i.e., $h^2=-2(d+a)k+(d+a)^2.$

The locus of P is therefore given by

$$x^2=-2(d+a)y+(d+a)^2$$

which is clearly a parabola.

[If the y-axis intersects the given circle and the given line in R and M and A be the mid-point of RM and $OA=p$, then

$$p=OR+RA=a+\frac{RM}{2}=a+\frac{d-a}{2}=\frac{d+a}{2}$$

so that the equation to the parabola becomes

$$x^2=-4py+4p^2,$$

i.e., $$x^2=-4p(y-p)$$

which represents a parabola which is concave downwards, whose vertex is at $A\ (o,\ p)$ and whose focus is at a distance p from the vertex A, *i.e.*, at the point O, the centre of the given circle.]

Problem 4. *To find the locus of a point which moves so that two of the three normals drawn from it to the parabola* $y^2=4ax$ *may be at right angles.*

The line $y=mx-2am-am^3$ is always a normal to the parabola $y^2=4ax$. If it passes through (h, k), we have

$$k=mh-2am-am^3,$$

i.e., $$am^3+m(2a-h)+k=0 \quad \ldots \quad \ldots \quad (1)$$

The three roots of this equation correspond to the three normals which pass through (h, k). If the three roots be m_1, m_2 and m_3, we have

$$m_1.m_2.m_3=-\frac{k}{a} \quad \ldots \quad \ldots \quad (2)$$

If two of the normals, say those corresponding to m_1 and m_2 be at right angles, we get

$$m_1m_2=-1, \text{ so that from (2), } m_3=\frac{k}{a}.$$

Since, m_3 is a root of the equation (1), $\frac{k}{a}$ satisfies this equation and hence,

$$a\cdot\frac{k^3}{a^3}+\frac{k}{a}(2a-h)+k=0,$$

i.e., $$k^2+a(2a-h)+a^2=0,$$

i.e., $$k^2=a(h-3a).$$

The locus of (h, k) is therefore given by

$$y^2=a(x-3a)$$

which represents a parabola whose vertex is at the point $(3a, 0)$ and whose latus rectum is one quarter of that of the original parabola.

EXERCISE VII(C)

1. Show that the chord $4x+3y+1=0$ of the parabola $y^2=8x$ is bisected at the point $(2, -3)$. [C. U.]

2. Find the equation to the chord of the parabola $y^2=8x$ which is bisected at the point $(3, 2)$.

3. Find the middle point of the chord of the parabola $y^2=8x$ whose equation is $3y=4x+1$.

4. Find the equation to the diameter of the parabola $y^2=12x$ which bisects the system of chords whose equation is $2x-3y+c=0$, where c varies.

5. Prove analytically that the tangents at the ends of any chord of a parabola meet on the diameter bisecting the chord.

6. Perpendiculars are drawn from the focus upon tangents to the parabola $y^2=4ax$; prove that the feet of these perpendiculars lie on the tangent at the vertex.

[**Hints :** Find the point of intersection of the lines $y=mx+\dfrac{a}{m}$ and $y=-\dfrac{1}{m}(x-a)$ and show that it satisfies the equation $x=0$.]

7. Prove that the tangents drawn at the extremities of any focal chord of a parabola intersect on the directrix. [*C. U.*]

8. Find the locus of the point of intersection of two tangents to the parabola $y^2=4ax$ when

(*i*) the sum of the gradients of the tangents is a constant quantity k;

(*ii*) the tangents meet at an angle $45°$.

9. Prove that the locus of the middle points of all chords of the parabola $y^2=4ax$ which are drawn through the vertex, is the parabola $y^2=2ax$.

10. Prove that the locus of the middle points of the focal chords of a parabola is another parabola whose vertex is at the focus of the given parabola and whose latus rectum is half the latus rectum of the given parabola.

11. Prove that the locus of the middle point of the portion of a normal of a parabola intercepted between the curve and the axis is a parabola whose vertex is the focus of the original parabola and whose latus rectum is one quarter of that of the original parabola. [*C. U.*]

12. From a point P, three normals are drawn to the parabola $y^2=4ax$. If two of them make angles with the axis of the parabola whose sum is $90°$, find the locus of P.

13. Find the locus of the middle points of normal chords of the parabola $y^2=4x$.

[**Hints :** If (h, k) be a point on the locus, then the chord whose middle point is (h, k) must be a normal to the parabola, so that the equation $y-k=\dfrac{2}{k}(x-h)$ is identical with $y=mx-2m-m^3$ (since $a=1$). Now compare coefficients and eliminate m.]

14. A point moves so that its distance from the straight line $ax+by+c=0$ is always equal to the tangent drawn from it to the circle $x^2+y^2=r^2$. Prove that the locus of the point is a parabola.

Answers :

2. $2x-y=4$. 3. $(2, 3)$. 4. $y=0$.

8. (*i*) $y=kx$; (*ii*) $y^2-4ax=(a+x)^2$.

12. $y^2=a(x-a)$. 13. $y^2(y^2-2x+4)+8=0$.

CHAPTER VIII

THE ELLIPSE

VIII-1. Definitions :

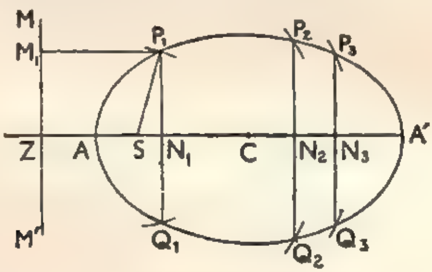
An ellipse is a conic section of which the eccentricity e is less than unity. We can therefore define it as follows :

An **Ellipse** is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point in the plane to its distance from a fixed straight line in the same plane is a constant quantity less than unity.

The fixed point is called the **Focus**, the fixed straight line is called the **Directrix** and the constant ratio is called the **Eccentricity** denoted by the letter e.

VIII-2. Construction of the curve :

To trace the ellipse when the directrix, the position of the focus and the eccentricity are given.



Let S be the focus and MM' the directrix and let e (<1) be the eccentricity.

Draw SZ perpendicular to MM'. Divide SZ internally at A and externally at A' in the ratio $e : 1$, so that $\frac{SA}{AZ}=e$, and $\frac{SA'}{A'Z}=e$, *i.e.*, $SA=e.AZ$ and $SA'=e.A'Z$. Hence, from the definition, A and A' are points on the ellipse.

Take a point N_1 on AA' and draw $P_1N_1Q_1$ perpendicular to AA'. With centre S and radius $e.ZN_1$ draw an arc cutting

$P_1N_1Q_1$ at P_1 and Q_1. Then P_1 and Q_1 are points on the ellipse, for joining SP_1 and drawing P_1M_1 perpendicular to MM', we have clearly,

$$SP_1 = e.ZN_1 \text{ (by construction)} = e.P_1M_1.$$

Similarly for Q_1.

Taking other points N_2, N_3 etc. we similarly obtain the points P_2, Q_2, P_3, Q_3 etc.

The curve drawn through these points is an ellipse.

VIII-3. Two important relations :

To prove that for an ellipse

$$(i)\ CA = e.CZ \text{ and } (ii)\ CS = e.CA,$$

where C is the middle point of AA'.

From the previous article, we have

$$SA = e.AZ \ \ldots (1) \quad \text{and } SA' = e.A'Z \quad \ldots \quad (2)$$

Adding, $\quad SA + SA' = e(AZ + A'Z)$,

or, $\quad AA' = e[(CZ - CA) + (CZ + CA')]$

i.e., $\quad 2CA = 2.e.CZ$, since $CA = CA'$.

$\therefore \quad$ **CA = e.CZ.** $\quad \ldots \quad \ldots \quad \ldots \quad (i)$

Subtracting (1) from (2),

$$SA' - SA = e(A'Z - AZ),$$

or, $\quad (CS + CA') - (CA - CS) = e.AA'$,

i.e., $\quad 2.CS = 2.e.CA.$

$\therefore \quad$ **CS = e.CA.** $\quad \ldots \quad \ldots \quad \ldots \quad (ii)$

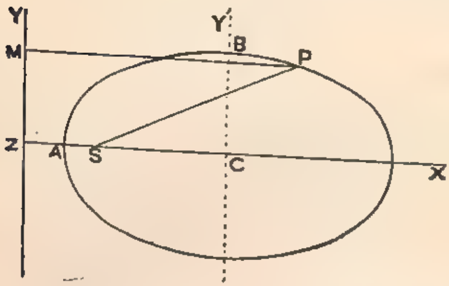
If the length AA' be $2a$, then the relations (i) and (ii) give

$$\mathbf{CZ = \frac{a}{e}} \quad \text{and} \quad \mathbf{CS = a\,e.}$$

VIII-4. Equation of the ellipse :

To find the equation to an ellipse referred to the directrix and the perpendicular from the focus upon the directrix as axes of coordinates.

Let S be the focus, ZM the directrix and e the eccentricty.



Draw SZ perpendicular to the directrix and produce ZS to any point X.

Let ZX, ZM be taken as the axes of x and y respectively.

Suppose that the length $ZS=d$, so that the point S is $(d, 0)$.

If $P(x, y)$ be any point on the locus, then the condition satisfied by P is

$SP=e.PM$, where PM is drawn perpendicular to ZM.

$\therefore \quad SP^2=e^2.PM^2$

i.e., $(x-d)^2+y^2=e^2x^2$

or, $\mathbf{x^2(1-e^2)+y^2-2dx+d^2=0}$

which is the required equation.

VIII-5. The standard equation :

The equation obtained in the previous article can be reduced to a simpler form by transferring the origin to a suitably chosen point.

We have $d=SZ=CZ-CS$

$$=\frac{a}{e}-ae \qquad \text{[Ref. Art. VIII-3]}$$

$$=\frac{a}{e}(1-e^2).$$

Substituting this value of d in the equation derived in the last article

viz. $\quad x^2(1-e^2)+y^2-2dx+d^2=0$

we get $\quad x^2(1-e^2)+y^2-2\frac{a}{e}(1-e^2)x+\frac{a^2}{e^2}(1-e^2)^2=0,$

$$i.e., \quad (1-e^2)\left\{x^2-2\frac{a}{e}x\right\}+y^2+\frac{a^2}{e^2}(1-e^2)^2=0$$

$$i.e., \quad (1-e^2)\left\{x^2-2\frac{a}{e}x+\frac{a^2}{e^2}\right\}+y^2=\frac{a^2}{e^2}(1-e^2)-\frac{a^2}{e^2}(1-e^2)^2$$

which reduces to

$$(1-e^2)\left(x-\frac{a}{e}\right)^2+y^2=a^2(1-e^2)$$

whence,
$$\frac{\left(x-\frac{a}{e}\right)^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1.$$

Referred to parallel axes through $C\left(\frac{a}{e}, 0\right)$ $\left[\because \quad CZ=\frac{a}{e}\right]$ the transformed equation is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1 \qquad \cdots \qquad \cdots \qquad (1)$$

If the curve meets the new axis of y i.e., CY' in B, and the length CB is denoted by b, then the coordinates $(0, b)$ must satisfy the equation (1).

Hence,
$$\frac{b^2}{a^2(1-e^2)}=1 \quad \text{or,} \quad b^2=a^2(1-e^2) \qquad \cdots \qquad (2)$$

The equation (1) then becomes

$$\mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

which is the equation to the curve referred to CX and the line through C parallel to the directrix as axes of coordinates.

This is the simplest form of the equation to an ellipse and is, therefore, taken as the **standard equation** to the curve.

We may, however, derive the standard equation independently thus:

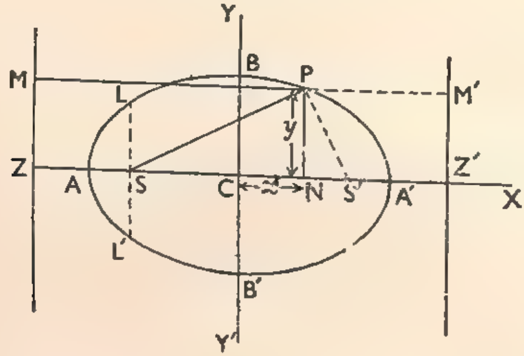
Let S be the focus, MZ the directrix and e the eccentricity. Draw SZ perpendicular to the directrix and produce ZS to any point X. If A and A' be points on ZS and ZS produced respectively such that

$$SA=e.AZ \quad \text{and} \quad SA'=e.A'Z$$

and if the length AA' be $2a$ and C the mid-point of AA', then we have

$$CS=ae \quad \text{and} \quad CZ=\frac{a}{e}.$$

We choose C as the origin, CX as the axis of x and CY, a line through C perpendicular to AA' as the axis of y.



Let $P(x, y)$ be any point on the curve. Draw the ordinate PN and also draw PM perpendicular to the directrix.

Now, the condition satisfied by P is

$$SP = e.PM.$$

$\therefore$ $$SP^2 = e^2PM^2 = e^2.ZN^2,$$

i.e., $$(x+ae)^2 + y^2 = e^2\left(x+\frac{a}{e}\right)^2,$$

i.e., $$x^2(1-e^2) + y^2 = a^2(1-e^2),$$

whence $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1;$$

i.e., $$\frac{\mathbf{x}^2}{\mathbf{a}^2} + \frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1},$$

where, as above, $b^2 = a^2(1-e^2)$.

This is the simplest form of the equation to an ellipse and is taken as the standard equation.

VIII-6. Geometrical property expressed by the standard equation :

The equation (3) of the previous article can be written as

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2},$$

or, $$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2},$$

i.e., $$\frac{y^2}{(a+x)(a-x)}=\frac{b^2}{a^2},$$

i.e., $$\frac{PN^2}{AN.A'N}=\frac{CB^2}{CA^2},$$

which may be stated as :

The square on the ordinate of any point on an ellipse varies as the rectangle contained by the segments of the major axis made by the ordinate.

VIII-7. Second focus and second directrix :

In the figure of *Art. VIII*-5 take S' and Z' on the positive side of the x-axis, such that

$$CS'=CS=ae, \quad \text{and} \quad CZ'=CZ=\frac{a}{e}.$$

Draw $Z'M'$ perpendicular to CZ' and PM' perpendicular to $Z'M'$ and join PS'.

The equation to the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1, \qquad [(1) \text{ of Art. VIII-5}]$$

or, $$x^2(1-e^2)+y^2=a^2(1-e^2),$$

i.e., $$x^2+a^2e^2+y^2=a^2+e^2x^2,$$

which can be written as

$$(a^2e^2-2aex+x^2)+y^2=a^2+e^2x^2-2aex,$$

i.e., $$(ae-x)^2+y^2=e^2\left(\frac{a}{e}-x\right)^2,$$

or, $$(CS'-CN)^2+PN^2=e^2(CZ'-CN)^2,$$

or, $$NS'^2+PN^2=e^2NZ'^2,$$

i.e., $$S'P^2=e^2PM'^2,$$

i.e., the distance of P from $S'=e\times$the distance of P from $Z'M'$. Hence, the same curve might have been described with S' as focus and $Z'M'$ as directrix. In other words, the ellipse has a second focus and a second directrix.

VIII-8. Shape of the curve :

Consider the standard equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

The equation can be written as

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}, \text{ or, } y=\pm\frac{b}{a}\sqrt{a^2-x^2}.$$

It follows that—

If $x^2>a^2$, *i.e.*, $(x+a)(x-a)>0$, *i.e.*, $x>a$, or, $<-a$. y is imaginary, showing that there is no point of the curve either to the right of the line $x=a$ or to the left of the line $x=-a$, the curve lying entirely between these two lines.

If $x^2=a^2$, *i.e.*, $x=a$, or, $-a$, we get two equal values of y namely zero. Hence, the lines $x=a$ and $x=-a$ are tangents at A' and A respectively.

If $x^2<a^2$, *i.e.*, $(x+a)(x-a)<0$, *i.e.*, x lies between $-a$ and a, we get two equal and opposite values of y, showing that the curve is symmetrical with respect to the axis of x.

Similarly, writing the equation in the form

$$x=\pm\frac{a}{b}\sqrt{b^2-y^2}$$

it can be seen that the curve lies entirely between the two lines $y=b$ and $y=-b$ which are respectively tangents to the curve at B and B' and is symmetrical with respect to the y-axis.

Also it is clear from the equation that as x increases in magnitude y decreases, and as y increases x decreases.

These facts are sufficient to enable us to form an idea as to the shape of the curve which is as shown in figure of *Art. VIII-2.*

It will be seen that unlike the parabola, the ellipse is a closed curve.

VIII-9. Definitions :

Vertices : The points A and A' where the line joining the foci meets the curve are called the vertices of the ellipse.

Centre : The middle point of AA', *i.e.*, C is called the centre of the ellipse.

Axes : The line AA' is called the **Major axis** and the line BB' is called the **Minor axis** of the ellipse.

Latus rectum : The double ordinate passing through the focus is called the latus rectum of the ellipse.

VIII-10. The eccentricity :

The relation

$$b^2=a^2(1-e^2), \quad \ldots \quad [\,(2) \textit{ of Art. VIII-5}\,]$$

connects the semi-major axis a, the semi-minor axis b and the

eccentricity e, from which any two of the quantities being given the third can be found out. In terms of a and b

$$\mathbf{e}=\frac{\mathbf{1}}{\mathbf{a}}\sqrt{\mathbf{a}^2-\mathbf{b}^2}.$$

VIII-11. The latus rectum (LSL′).

If SL be the ordinate corresponding to the focus S, then the coordinates of the point L are $(-ae, SL)$.

Hence, $\frac{a^2e^2}{a^2}+\frac{SL^2}{b^2}=1.$

$$\therefore \quad SL^2=b^2(1-e^2)=b^2\cdot\frac{b^2}{a^2}.$$

$\therefore$ The semi-latus rectum $SL=\frac{b^2}{a}.$

Hence, the latus rectum $\mathbf{LSL'}=\frac{\mathbf{2b^2}}{\mathbf{a}}.$

VIII-12. An important property :

To prove that the sum of the focal distances of any point on an ellipse is constant and equal to the major axis.

If $P(x, y)$ is any point on the curve (*fig. Art. VIII-5*),

$$SP=e.PM=e.ZN=e(CZ+CN)=e\left(\frac{a}{e}+x\right)=a+ex.$$

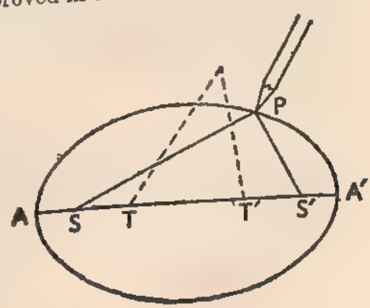
$$S'P=e.PM'=e.NZ'=e(CZ'-CN)=e\left(\frac{a}{e}-x\right)=a-ex.$$

Hence, $\quad SP+S'P=(a+ex)+(a-ex)=2a$

$\qquad\qquad\qquad\qquad =$ the major axis AA'.

VIII-13. Mechanical construction :

From the property of the ellipse proved in the last article, we get an easy method for the mechanical construction of the curve.



We take a piece of string of length equal to the major axis $2a$ of the ellipse and fasten its extremities at S and S' the positions of the foci of the ellipse. The string is now kept tightly stretched by means of a pencil which is placed vertically against the string. The point

P of the pencil as it moves about on the paper will describe a curve which is an ellipse, for

$$SP+S'P = \text{length of the string} = \text{the major axis.}$$

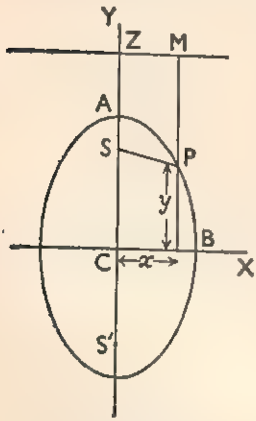
Note : If on SS' we take two points T and T' such that $ST=S'T'$ and take these as the positions of the foci we shall get an ellipse of different shape but of the same major axis AA'.

VIII-14. Major axis along the axis of y :

If the major axis of the ellipse is taken along the y-axis, then its equation will be deduced from the standard equation by simply interchanging x and y, and the resulting equation is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

It can also be derived independently thus : The focus S in this case is the point $(0, ae)$.



The condition $SP^2=e^2PM^2$ now gives

$$(x-0)^2+(y-ae)^2=e^2\left(\frac{a}{e}-y\right)^2$$ which reduces to

$$\frac{x^2}{a^2(1-e^2)}+\frac{y^2}{a^2}=1,$$

i.e., $$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, \text{ where}$$

$$b^2=a^2(1-e^2).$$

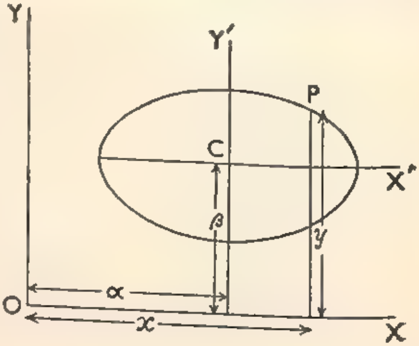
Note : *Here the foci are* $(0, \pm ae)$ *and the equations to the directrices are* $y=\pm\frac{a}{e}$.

VIII-15. Axes parallel to the axes of coordinates :

Let the centre of the ellipse be at the point (α, β), the major axis of length $2a$ parallel to the x-axis and the minor axis of length $2b$ parallel to the y-axis.

Clearly, with CX' and CY' as axes of coordinates, the equation to the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Now, the point C being (α, β), the coordinates of O referred to CX' and CY' as axes are $(-\alpha, -\beta)$. Therefore, to get the equation to the ellipse with reference to OX and OY as axes of coordinates, we have simply to replace x by $x-\alpha$ and y by $y-\beta$ and the resulting equation is

$$\mathbf{\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1.}$$

This, therefore, is the required equation to the ellipse whose centre is at (α, β) and whose axes are parallel to the axes of coordinates.

Simplifying, the last equation can be put in the form

$$lx^2+my^2+2gx+2fy+c=0,$$

where l and m are two different positive quantities.

Hence also, given an equation of the form

$$lx^2+my^2+2gx+2fy+c=0,$$

we at once conclude that it represents an ellipse whose axes are parallel to the axes of coordinates.

VIII-16. General equation :

When the position of the focus and the equation of the corresponding directrix are given with reference to any set of rectangular axes, the equation of the ellipse can be easily found out when its eccentricity e is known.

If $S(\alpha, \beta)$ be the focus and $Ax+By+C=0$ be the directrix, then any point $P(X, Y)$ on the ellipse must satisfy the condition $SP=e.PM$ where PM is the perpendicular from (X, Y) upon $Ax+By+C=0$. We then have,

$$(X-\alpha)^2+(Y-\beta)^2=e^2\left\{\frac{AX+BY+C}{\sqrt{A^2+B^2}}\right\}^2.$$

The locus of (X, Y) is therefore given by

$$(A^2+B^2)\{(x-\alpha)^2+(y-\beta)^2\}=e^2(Ax+By+C)^2$$

which is the required equation of the ellipse.

The last equation can be reduced to the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

It can be seen from particular examples that the condition $\mathbf{h^2 < ab}$ is always satisfied whenever the locus represented by the equation is an ellipse.

VIII-17. Position of a point $(\mathbf{x_1, y_1})$ with respect to the ellipse $\dfrac{\mathbf{x^2}}{\mathbf{a^2}}+\dfrac{\mathbf{y^2}}{\mathbf{b^2}}=\mathbf{1}$.

[Draw a figure]

Let P be the point (x_1, y_1) and let PN drawn perpendicular to the major axis meet the curve in Q.

Since, the point Q whose coordinates are (x_1, QN) is on the ellipse, we have

$$\frac{x_1^2}{a^2}+\frac{QN^2}{b^2}=1, \text{ whence } QN^2=b^2\left(1-\frac{x_1^2}{a^2}\right)$$

Clearly, the point $P(x_1, y_1)$ is outside, on or inside the ellipse according as

$$PN > = \text{ or, } < QN$$

i.e., according as $PN^2 > = \text{ or, } < QN^2$

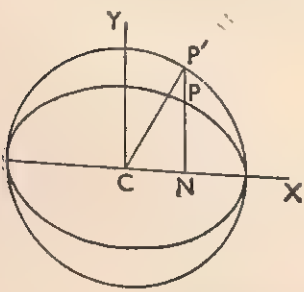
i.e., according as $y_1^2 > = \text{ or, } < b^2\left(1-\dfrac{x_1^2}{a^2}\right)$

i.e., according as the expression $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1$ is positive, zero or negative.

Note : The same result will be arrived at if P is compared with a point on the curve having the same ordinate as the point P.

VIII-18. Auxiliary circle :

The circle described on the major axis of an ellipse as diameter is called the auxiliary circle. Its equation is clearly $x^2+y^2=a^2$.



Let a point P be taken on the ellipse and the ordinate PN produced to meet the auxiliary circle in P'. CP' is joined.

Now, P' being a point on the circle,

$$CN^2+P'N^2=a^2.$$

$$\therefore \quad P'N = \sqrt{a^2 - CN^2} \quad \ldots \quad \ldots \quad (1)$$

Again, P being a point on the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ we have}$$

$$\frac{CN^2}{a^2} + \frac{PN^2}{b^2} = 1.$$

$$\therefore \quad PN = \frac{b}{a}\sqrt{a^2 - CN^2} \quad \ldots \quad (2)$$

Hence, from (1) and (2),

$$PN = \frac{b}{a}.\ P'N,$$

which gives the relation between the ordinates of the points P and P' which are called **corresponding points.**

VIII-19. Eccentric angle :

If the angle NCP' be denoted by ϕ, then

$$CN = a \cos\phi \text{ and } P'N = a \sin\phi.$$

$$\therefore \quad PN = \frac{b}{a}P'N = \frac{b}{a}.\ a \sin\phi = b \sin\phi.$$

Hence, the coordinates of any point P on the ellipse, expressed in terms of a single variable ϕ, are

$$\mathbf{a \cos\phi \text{ and } b \sin\phi}$$

The angle ϕ is called the **eccentric angle** of the point P. It is the angle which the radius through the corresponding point on the auxiliary circle makes with the major axis.

WORKED OUT EXAMPLES

Ex. 1. *Find the foci and directrices of the ellipse,*

$$9x^2 + 4y^2 = 36.$$

Written in the standard form, the equation becomes

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

The major axis of the ellipse therefore lies along the axis of y. [*Art. VIII-14*]

The eccentricity e is obtained from the relation

$$4=9(1-e^2).$$

$$\therefore \qquad e=\frac{\sqrt{5}}{3}.$$

Since, the semi-major axis is of length 3, the coordinates of the foci are

$$\left(0,\ \pm 3\frac{\sqrt{5}}{3}\right),$$

i.e.,

$$(0,\ \pm\sqrt{5}).$$

The equations of the directrices are

$$y=\pm\frac{3}{\frac{\sqrt{5}}{3}},$$

i.e.,

$$y=\pm\frac{9}{\sqrt{5}}.$$

Ex. 2. *Find the coordinates of the centre, the eccentricity and the latus rectum of the ellipse,*

$$x^2+2y^2+2x-8y+2=0.$$

To get the centre, we express the equation in the form

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1. \qquad [\textit{ Art. VIII-15 }]$$

The equation is written as

$$(x^2+2x+1)+2(y^2-4y+4)=1+8-2,$$

i.e.,

$$(x+1)^2+2(y-2)^2=7,$$

or,

$$\frac{(x+1)^2}{7}+\frac{(y-2)^2}{\frac{7}{2}}=1.$$

The centre is therefore the point $(-1, 2)$.

If parallels to the axes through this point are taken as axes of reference, the equation to the ellipse reduces to

$$\frac{x^2}{7}+\frac{y^2}{\frac{7}{2}}=1.$$

Hence, $a=$ the semi-major axis $=\sqrt{7}$

and $b=$ the semi-minor axis $=\sqrt{\frac{7}{2}}$.

$\therefore$ The latus rectum $=\dfrac{2b^2}{a}=\dfrac{2\times\frac{7}{2}}{\sqrt{7}}=\sqrt{7}.$

The eccentricity is given by $b^2=a^2(1-e^2)$,

i.e., $\frac{7}{2}=7(1-e^2)$ giving $e=\dfrac{\ }{\sqrt{2}}.$

Ex. 3. *Find the equation to the ellipse whose latus rectum is* $\frac{4}{3}$ *and whose eccentricity is* $\frac{1}{\sqrt{3}}$, *the centre being at the point* $(2, -1)$ *and major axis parallel to the axis of* y.

If the semi-major and semi-minor axes be respectively a and b, we have

$$\frac{2b^2}{a}=\frac{4}{3} \quad \cdots \quad (1) \qquad \text{and} \qquad \left(\frac{1}{\sqrt{3}}\right)^2=\frac{a^2-b^2}{a^2} \quad \cdots \quad (2)$$

From (2), $3a^2-3b^2=a^2$,

i.e., $2a^2=3b^2$

$\therefore$ $2a^2=2a$ *from* (1)

$\therefore$ $a=1$, $b=\sqrt{\frac{2}{3}}$.

The equation to the ellipse having centre at the origin and major axis along the axis of y is clearly

$$\frac{x^2}{\frac{2}{3}}+\frac{y^2}{1}=1.$$

$\therefore$ The required equation to the ellipse having centre at $(2, -1)$ and major axis parallel to the axis of y is

$$\frac{(x-2)^2}{\frac{2}{3}}+\frac{(y+1)^2}{1}=1,$$

i.e., $$3(x-2)^2+2(y+1)^2=2,$$

i.e., $$3x^2+2y^2-12x+4y+12=0.$$

Ex. 4. *Find the equation of the ellipse (referred to its axes as the axes of* x *and* y *respectively) which passes through the point* $(-3, 1)$ *and has the eccentricity* $\sqrt{\frac{2}{5}}$. [*C. U.*]

Let the required equation be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Since, it passes through $(-3, 1)$, we have

$$\frac{9}{a^2}+\frac{1}{b^2}=1 \quad \cdots \quad \cdots \quad (1)$$

Also from the relation $b^2=a^2(1-e^2)$, we get

$$b^2=a^2(1-\tfrac{2}{5}),$$

i.e., $$b^2=\tfrac{3}{5}a^2. \qquad \dots \qquad \dots \quad (2)$$

From (1) and (2),

$$\frac{9}{a^2}+\frac{5}{3a^2}=1.$$

$\therefore$ $$a^2=\tfrac{32}{3} \text{ and } b^2=\tfrac{32}{5}.$$

The required equation is therefore, $3x^2+5y^2=32$.

Ex. 5. *A point moves so that the sum of its distances from two fixed points is constant. Prove that the locus of the point is an ellipse.*

Let the line joining the two fixed points be taken as the x-axis and the mid-point of the join as the origin, and let the coordinates of the points be $(-a, 0)$ and $(a, 0)$.

If then (x, y) be a point on the locus, we have, from the given condition,

$$\sqrt{(x+a)^2+y^2}+\sqrt{(x-a)^2+y^2}=\text{constant}=2k \text{ (say)}.$$

i.e., $$\sqrt{(x+a)^2+y^2}=2k-\sqrt{(x-a)^2+y^2}.$$

Squaring, $(x+a)^2+y^2=4k^2+(x-a)^2+y^2-4k\sqrt{(x-a)^2+y^2}$ which reduces to

$$ax-k^2=-k\sqrt{(x-a)^2+y^2}.$$

Squaring again, we get

$$k^4-2k^2ax+a^2x^2=k^2(x-a)^2+k^2y^2,$$

i.e., $$x^2(k^2-a^2)+k^2y^2=k^2(k^2-a^2),$$

or, $$\frac{x^2}{k^2}+\frac{y^2}{k^2-a^2}=1$$

as the required equation to the locus.

Since, $2k$ is necessarily greater than the distance between the given points *i.e.*, $2a$, the quantity k^2-a^2 is positive and hence the locus is an ellipse whose major axis of length $2k$ lies along the line joining the given points.

EXERCISE VIII (A)

1. Find the eccentricity of the ellipse of which the major axis is double the minor axis.

2. If the minor axis of an ellipse is equal to the distance between its foci, prove that its eccentricity is $\frac{1}{\sqrt{2}}$.

3. Find the latus rectum and eccentricity of the ellipse whose semi-axes are 5 and 4.

4. Find the equation to the ellipse referred to its axes as axes of coordinates, which passes through the points (2, 3) and (−4, 1).

5. Find the equation to the ellipse referred to its centre as origin whose axes are 10 and 8, when the major axis is

(*i*) along the axis of x ;

(*ii*) along the axis of y.

6. Find the equation to the ellipse whose centre is at (−2, 3) and whose semi-axes are 3 and 2, when the major axis is

(*i*) parallel to the axis of x ;

(*ii*) parallel to the axis of y.

7. Find the equation of the ellipse referred to its axes as axes of co-ordinates, whose

(*i*) major axis is $\frac{9}{2}$ and eccentricity is $\frac{1}{\sqrt{3}}$;

(*ii*) latus rectum is 5 and eccentricity is $\frac{2}{3}$. [*C. U.*]

8. The distance between the foci of an ellipse is 10 and its latus rectum is 15 ; find its equation referred to its axes as axes of coordinates.

9. Find the equation to the ellipse, whose

(*i*) focus is (2, 1), directrix is $2x-y+3=0$ and eccentricity is $\frac{1}{\sqrt{2}}$;

(*ii*) focus is (−1, 1), directrix is $x-y+3=0$ and eccentricity is $\frac{1}{2}$. [*C. U.* 1952]

10. Find the latus rectum, eccentricity and coordinates of the foci of the following ellipses :

(*i*) $4x^2+5y^2=1$; (*ii*) $16x^2+9y^2=144$;

(*iii*) $4x^2+3y^2-6y-9=0$; (*iv*) $3x^2+4y^2-6x+16y-89=0$.

11. Find the distance between a focus and an extremity of the minor axis of the ellipse :

(*i*) $4x^2+5y^2=100$; (*ii*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

12. Find the equation to the ellipse whose centre is at (0, 2) and major axis along the axis of y, whose minor axis is equal to the distance between the foci and whose latus rectum is 2.

13. Find the equation to the ellipse whose eccentricity is $\frac{2}{3}$, focus S is (3, 0) and vertex A is (1, 0).

14. The distance between two fixed points A and B is 4. A point P moves so that $PA+PB=6$. Find the locus of P.

15. The distance of a point on the conic $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2. Find the eccentric angle. [C. U.]

16. Find the eccentric angles of the positive extremities of the latera recta of the ellipse $3x^2+4y^2=48$.

17. The ordinate of a point P on an ellipse whose centre is O and semi-minor axis b is produced to meet the auxiliary circle at Q. The parallel to OQ through P meets the major axis in G. Prove that $PG=b$. [C. U.]

18. Prove that the equation to the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ joining two points whose eccentric angles are $\theta+\phi$ and $\theta-\phi$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\cos\phi.$$

Answers :

1. $\frac{\sqrt{3}}{2}$. 3. latus rectum $=\frac{32}{5}$, $e=\frac{3}{5}$.

4. $2x^2+3y^2=35$. 5. (i) $\frac{x^2}{25}+\frac{y^2}{16}=1$; (ii) $\frac{x^2}{16}+\frac{y^2}{25}=1$.

6. (i) $4x^2+9y^2+16x-54y+61=0$; (ii) $9x^2+4y^2+36x-24y+36=0$.

7. (i) $16x^2+24y^2=81$; (ii) $20x^2+36y^2=405$.

8. $3x^2+4y^2=300$. 9. (i) $6x^2+9y^2+4xy-52x-14y+41=0$.
(ii) $7x^2+7y^2+2xy+10x-10y+7=0$.

10. (i) latus rectum $\frac{4}{5}$, $e=\frac{1}{\sqrt{5}}$, foci $\left(\pm\frac{}{2\sqrt{5}}, \frac{0}{}\right)$;
(ii) latus rectum $\frac{9}{2}$, $e=\frac{\sqrt{7}}{4}$, foci $(0, \pm\sqrt{7})$;
(iii) latus rectum 3, $e=\frac{1}{2}$, foci (0, 0) and (0, 2) ;
(iv) latus rectum 9, $e=\frac{1}{2}$, foci $(-2, -2)$ and $(4, -2)$.

11. (i) 5 ; (ii) a. 12. $2x^2+y^2-4y=0$.

13. $5x^2+9y^2-70x+65=0$. 14. $\frac{x^2}{9}+\frac{y^2}{5}=1$.

15. $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$ or $\frac{7\pi}{4}$. 16. $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

VIII-20. Tangent at a point :

To find the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the point* (x_1, y_1).

Let P be the given point (x_1, y_1). We take a point Q on the ellipse close to P. Let its coordinates be (x_2, y_2).

The equation to PQ is $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$.

But since P, Q are points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad \text{and} \quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1$$

whence $$\frac{x_2^2-x_1^2}{a^2}+\frac{y_2^2-y_1^2}{b^2}=0$$

i.e., $$\frac{(x_2-x_1)(x_2+x_1)}{a^2}+\frac{(y_2-y_1)(y_2+y_1)}{b^2}=0$$

$\therefore$ $$\frac{y_2-y_1}{x_2-x_1}=-\frac{b^2(x_2+x_1)}{a^2(y_2+y_1)}.$$

The *equation to the chord PQ* of the ellipse is therefore

$$y-y_1=-\frac{b^2(x_2+x_1)}{a^2(y_2+y_1)}(x-x_1)$$

If now, Q tends to coincidence with P so that x_2 tends to the value x_1 and y_2 to y_1, the chord in this limiting position becomes the tangent at P. The required equation of the tangent, obtained by putting $x_2=x_1$ and $y_2=y_1$ is thus

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

i.e., $$(x-x_1)\frac{x_1}{a^2}+(y-y_1)\frac{y_1}{b^2}=0$$

i.e., $$\mathbf{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}, \qquad \text{since } \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

Alternative method :

If (x, y) and $(x+\delta x, y+\delta y)$ are two close points on the curve, then

$$\frac{(x+\delta x)^2}{a^2}+\frac{(y+\delta y)^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

On subtraction, $$\frac{\delta x(2x+\delta x)}{a^2}+\frac{\delta y(2y+\delta y)}{b^2}=0.$$

Hence $$\frac{\delta y}{\delta x}=-\frac{b^2}{a^2}\,\frac{2x+\delta x}{2y+\delta y}.$$

$$\therefore\ \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(-\frac{b^2}{a^2}\cdot\frac{2x+\delta x}{2y+\delta y}\right)=-\frac{b^2x}{a^2y}$$

since, as δx approaches zero, δy also approaches zero.

The gradient of the tangent at (x_1, y_1) is therefore $-\frac{b^2x_1}{a^2y_1}$.

The required equation of the tangent is then

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

whence the equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \text{ follows.}$$

Note : Observe that the rule to write down the equation of the tangent to the circle applies here too.

VIII-21. Condition for tangency :

To find the condition that the straight line $y=mx+c$ ⋯ (1)

should touch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ⋯ ⋯ (2)

The abscissæ of the points in which the line (1) meets the curve (2) are the roots of the equation

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

i.e., $(a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$ ⋯ (3)

If the line is a tangent, the two points of intersection coincide and hence the equation (3) must have equal roots. The condition for this is

$$4a^4m^2c^2=4a^2(c^2-b^2)(a^2m^2+b^2)$$

which, on reduction, gives

$$c^2=a^2m^2+b^2,$$

i.e., $\mathbf{c=\pm\sqrt{a^2m^2+b^2}}$ ⋯ ⋯ (4)

This then is the required condition.

Remark : The equation (3) being quadratic in x, we conclude that the line always meets the ellipse in two points. It can be shown as in the case of the circle and the parabola that these are two real and distinct points if $c^2<a^2m^2+b^2$ and two imaginary points (*i.e.,* geometrically the line does not intersect the curve at all) if $c^2>a^2m^2+b^2$.

Note : 1. If $b=a$, the ellipse becomes a circle of radius a and the condition for tangency reduces to $c=\pm a\sqrt{1+m^2}$, as has already been derived in *Art. VI-12.*

Note : 2. For the existence of an infinite root of the equation (3) we should have $a^2m^2+b^2=0$, *i.e.,* $m=\pm\sqrt{\dfrac{-b^2}{a^2}}$ which is imaginary. We therefore conclude that no real line can meet the curve at an infinite distance from the origin, which is also otherwise evident from the fact that the ellipse is a closed curve.

VIII-22. Tangents in a given direction :

If the gradient m of a line is given, we find there are two values of c *viz.,* those obtained in (4) of *Art. VIII-21*, which

will make $y=mx+c$ a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Hence, corresponding to a given gradient m there are always two parallel tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose equations are

$$\mathbf{y=mx\pm\sqrt{a^2m^2+b^2}}.$$

VIII-23. Point of contact :

To find the point where the line $y=mx+\sqrt{a^2m^2+b^2}$ *touches the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the line can be written as

$$mx-y+\sqrt{a^2m^2+b^2}=0 \qquad \cdots \qquad \cdots \quad (1)$$

If the line (1) touches the ellipse at (x_1, y_1) then it must represent the same line as

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

i.e.,

$$b^2x_1.x+a^2y_1.y-a^2b^2=0 \qquad \cdots \qquad \cdots \quad (2)$$

Hence, from (1) and (2), comparing coefficients,

$$\frac{b^2x_1}{m}=\frac{a^2y_1}{-1}=\frac{-a^2b^2}{\sqrt{a^2m^2+b^2}}$$

$$\therefore \quad x_1=-\frac{a^2m}{\sqrt{a^2m^2+b^2}} \quad \text{and} \quad y_1=\frac{b^2}{\sqrt{a^2m^2+b^2}}$$

which give the coordinates of the required point of contact.

VIII-24. Number of tangents from a point :

To prove that two tangents, real or imaginary, can be drawn from a point to an ellipse.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let (x_1, y_1) be the given point.

$$\cdots \qquad \cdots \quad (1)$$

The line $y=mx+\sqrt{a^2m^2+b^2}$ is always a tangent to the ellipse.

We are now to choose m for which the line (1) may pass through (x_1, y_1). This requires

$$y_1 = mx_1 + \sqrt{a^2m^2+b^2},$$

i.e., $$(y_1 - mx_1)^2 = a^2m^2 + b^2,$$

i.e., $$(x_1{}^2 - a^2)m^2 - 2x_1y_1m + y_1{}^2 - b^2 = 0. \qquad \cdots \quad (2)$$

The equation being quadratic in m gives two values of m corresponding to each of which we have a tangent which passes through (x_1, y_1).

Hence, two tangents can be drawn from the given point to the ellipse and these are

$$\mathbf{y - y_1 = m_1(x - x_1)}$$

and $$\mathbf{y - y_1 = m_2(x - x_1)}$$

where m_1 and m_2 are the two roots of the equation (2).

The discriminant of equation (2) is

$$\begin{aligned} &4x_1{}^2y_1{}^2 - 4(x_1{}^2 - a^2)(y_1{}^2 - b^2) \\ &= 4(b^2x_1{}^2 + a^2y_1{}^2 - a^2b^2) \\ &= 4a^2b^2\left(\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} - 1\right). \end{aligned}$$

(*i*) If the disc. is positive, *i.e.*, $\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} - 1 > 0$, in which case the point lies outside the ellipse, we get two real and different roots and hence two distinct tangents can be drawn.

(*ii*) If the disc. is zero, *i.e.*, $\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} - 1 = 0$, *i.e.*, the point lies on the ellipse, we get two equal roots and hence the two tangents become coincident.

(*iii*) If the disc. is negative, *i.e.*, $\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} - 1 < 0$, which is the case when the point lies inside the ellipse, we get two imaginary roots and in this case both the tangents are imaginary, *i.e.*, geometrically no tangents can be drawn.

VIII-25. Normal at a point :

To find the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *at the point* (x_1, y_1).

The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

i.e., $$y=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1} \qquad \ldots \qquad \ldots \quad (1)$$

$\therefore$ The equation of the normal which is a straight line through (x_1, y_1) perpendicular to the tangent (1) is

$$y-y_1=m(x-x_1) \qquad \ldots \qquad \ldots \quad (2)$$

where $$m\times\left(-\frac{b^2x_1}{a^2y_1}\right)=-1,$$

i.e., $$m=\frac{a^2y_1}{b^2x_1}.$$

Hence, substituting for m in (2), the required equation is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

i.e., $$y-y_1=\frac{y_1/b^2}{x_1/a^2}(x-x_1),$$

i.e., $$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}.$$

VIII-26. Chord of contact :

To find the equation of the chord of contact of tangents drawn from the point (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

As in *Art. VI-19* if $T_1(\alpha_1, \beta_1)$ and $T_2(\alpha_2, \beta_2)$ be the points of contact of tangents drawn from the given point $P(x_1, y_1)$, then the condition that the tangents at T_1 and T_2 may pass through P, gives

$$\frac{x_1\alpha_1}{a^2}+\frac{y_1\beta_1}{b^2}=1 \text{ and } \frac{x_1\alpha_2}{a^2}+\frac{y_1\beta_2}{b^2}=1$$

which show that both the points (α_1, β_1) and (α_2, β_2) lie on the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

which is therefore the required equation of the chord of contact.

VIII-27. Diameter of the ellipse :

To find the locus of the middle points of a system of parallel chords of an ellipse.

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \cdots \qquad \cdots \quad (1)$$

and let m be the gradient of the system of parallel chords, so that the equation of any one of the system is

$$y=mx+c \qquad \cdots \qquad \cdots \quad (2)$$

where different values of c give different chords of the system.

If (x_1, y_1) and (x_2, y_2) be the points in which the line (2) meets the ellipse (1), then x_1 and x_2 must be the roots of the equation,

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

i.e., $\quad (a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0 \qquad \cdots \quad (3)$

Similarly, y_1 and y_2 are the roots of the quadratic in y obtained by eliminating x between the equations (1) and (2), *viz.*,

$$\frac{(y-c)^2}{m^2a^2}+\frac{y^2}{b^2}=1,$$

i.e., $\quad (a^2m^2+b^2)y^2-2b^2cy+b^2(c^2-a^2m^2)=0. \qquad \cdots \quad (4)$

If therefore (h, k) be the middle point of the chord, we have

$$h=\frac{x_1+x_2}{2}=-\frac{a^2mc}{a^2m^2+b^2}, \qquad \textit{from } (3)$$

and

$$k=\frac{y_1+y_2}{2}=\frac{b^2c}{a^2m^2+b^2}, \qquad \textit{from } (4).$$

On division, $\qquad \dfrac{k}{h}=-\dfrac{b^2}{a^2m},$

a relation independent of c and therefore true for any value of c, that is, for any position of the chord.

Hence, the required locus of (h, k) is given by

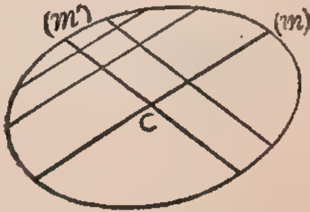
$$\mathbf{y}=-\frac{\mathbf{b}^2}{\mathbf{a}^2\mathbf{m}}\mathbf{x}$$

which is clearly a line passing through the centre of the ellipse.

Def. : *The locus of the middle points of a system of parallel chords of an ellipse is called a* **diameter.**

VIII-28. Conjugate diameters :

Def. : *Two diameters are said to be conjugate when each bisects all chords parallel to the other.*



From the previous article, the diameter bisecting chords parallel to $y=mx$ is

$$y=m'x$$

where, $m'=-\dfrac{b^2}{a^2m}$,

i.e., $mm'=-\dfrac{b^2}{a^2}$ ... ... (1)

Again, the diameter bisecting chords parallel to $y=m'x$ is

$$y=-\frac{b^2}{a^2m'}\,x,$$

i.e., $y=mx$, *from* (1)

We therefore, find that if (1) holds, then the two lines $y=mx$ and $y=m'x$ are such that each bisects all chords parallel to the other.

Hence, the condition that two diameters $y=mx$ and $y=m'x$ may be conjugate is

$$\mathbf{mm'=-\frac{b^2}{a^2}}.$$

VIII-29. Chord having its middle point given :

To find the equation of the chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, *which is bisected at a given point* (α, β).

Since, the chord passes through (α, β) let its equation be

$$y-\beta=m(x-\alpha) \qquad \dots \qquad \dots \qquad (1)$$

Now, (α, β) being the middle point of the chord must lie on $y=-\dfrac{b^2}{a^2m}x$ which is the diameter bisecting chords parallel to (1).

$\therefore \quad \beta = -\dfrac{b^2}{a^2 m}\alpha$ whence $m = -\dfrac{b^2\alpha}{a^2\beta}$.

$\therefore$ Substituting for m in (1), the required equation of the chord is

$$y-\beta = -\frac{b^2\alpha}{a^2\beta}(x-\alpha),$$

i.e.,
$$(\mathbf{x}-\alpha)\frac{\alpha}{\mathbf{a}^2}+(\mathbf{y}-\beta)\frac{\beta}{\mathbf{b}^2}=0.$$

VIII-30. Director Circle :

To find the locus of the point of intersection of a pair of tangents to an ellipse which meet at right angles.

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \ldots \qquad \ldots \quad (1)$$

If $P\,(h, k)$ be any point on the locus then the two tangents to (1), which pass through P must be at right angles.

Now, any tangent to (1) is given by

$$y = mx + \sqrt{a^2m^2+b^2}.$$

If it passes through $P\,(h, k)$, then

$$k = mh + \sqrt{a^2m^2+b^2},$$

$\therefore \qquad (k-mh)^2 = a^2m^2+b^2,$

i.e., $\qquad (h^2-a^2)m^2-2hkm+k^2-b^2=0 \qquad \ldots \quad (2)$

The roots m_1 and m_2 of the last equation give the gradients of the two tangents which pass through P.

Since, the tangents are at right angles, we have

$$m_1m_2=-1.$$

Hence, from the equation (2), $\dfrac{k^2-b^2}{h^2-a^2}=-1,$

i.e., $\qquad h^2+k^2=a^2+b^2.$

$\therefore$ (h, k) satisfies the equation

$$\mathbf{x}^2+\mathbf{y}^2=\mathbf{a}^2+\mathbf{b}^2$$

which is therefore the required equation to the locus.

The locus is clearly a circle having its centre at the centre of the ellipse. The circle is called the **Director Circle**.

VIII-31. Some geometrical properties :

1. *The tangent to an ellipse at the extremity of any diameter is parallel to the system of chords bisected by the diameter.*

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let the equation to any chord of it be

$$y=mx+c. \qquad \dots \qquad \dots \quad (1)$$

Let (x_1, y_1) be the point on the ellipse the tangent at which is parallel to (1).

We have then to prove that (x_1, y_1) lies on the diameter which bisects the system of chords parallel to (1).

The tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \qquad \dots \qquad \dots \quad (2)$$

Since, (1) and (2) are parallel, we have

$$m=-\frac{b^2x_1}{a^2y_1}.$$

The relation shows that (x_1, y_1) lies on the locus given by

$$m=-\frac{b^2x}{a^2y},$$

i.e.,
$$y=-\frac{b^2}{a^2m}x,$$

which is clearly the diameter bisecting chords parallel to (1).

2. *The tangents at the extremities of any chord of an ellipse meet on the diameter which bisects the chord.*

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$ and let a chord of the ellipse be

$$y=mx+c. \qquad \dots \qquad \dots \quad (1)$$

If P and Q be the extremities of the chord, the tangents at which to the ellipse meet at $T(x_1, y_1)$, then PQ is the chord of contact of tangents drawn from T and hence its equation is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \qquad \dots \qquad \dots \quad (2)$$

Since (1) and (2) represent the same straight line, we must have

$$m = -\frac{b^2 x_1}{a^2 y_1}.$$

Hence, (x_1, y_1) the point of intersection of the tangents lies on the straight line

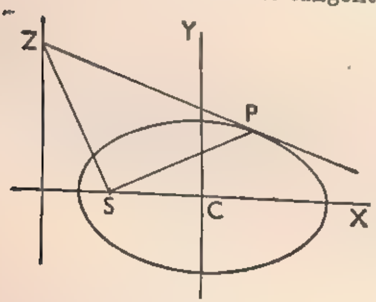
$$y = -\frac{b^2}{a^2 m} x.$$

But this is the diameter bisecting PQ and chords parallel to it. Hence the proposition.

3. *The portion of the tangent at any point of an ellipse intercepted between the point of contact and the directrix subtends a right angle at the focus.*

Let $P(x_1, y_1)$ be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

The equation of the tangent at P is



$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \quad \cdots \quad (1)$$

The equation of the directrix is

$$x = -\frac{a}{e} \quad \cdots \quad (2)$$

The point Z where (1) and (2) intersect is found by solving (1) and (2).

Substituting for x from (2) in (1), we have

$$-\frac{x_1}{ae}+\frac{yy_1}{b^2}=1,$$

i.e., $$y = \frac{b^2(ae+x_1)}{aey_1}$$

$\therefore$ The point Z has coordinates $-\frac{a}{e}$ and $\frac{b^2(ae+x_1)}{aey_1}$.

The gradient of $SZ = \dfrac{\dfrac{b^2(ae+x_1)}{aey_1}-0}{-\dfrac{a}{e}+ae}$,

(since S has coordinates $-ae$ and 0)

$$= -\frac{ae+x_1}{y_1}, \text{ on simplification.}$$

The gradient of SP is $\dfrac{y_1-0}{x_1+ae}$ *i.e.*, $\dfrac{y_1}{x_1+ae}$.

The product of the gradients $= -\dfrac{ae+x_1}{y_1} \times \dfrac{y_1}{x_1+ae} = -1$.

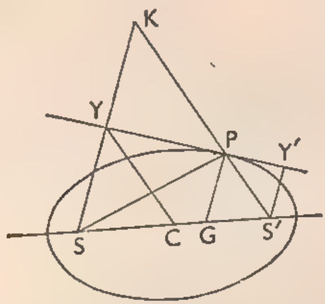
Hence, PSZ is a right angle, which proves the proposition.

4. *The normal at any point of an ellipse bisects the angle between the focal distances of the point.*

Let $P(x_1, y_1)$ be a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

We have $SP=a+ex_1$ and $S'P=a-ex_1$. ... (1)

[*Art. VIII-12*]



The equation of PG, the normal at P is

$$\frac{x-x_1}{\dfrac{x_1}{a^2}}=\frac{y-y_1}{\dfrac{y_1}{b^2}} \qquad \dots \quad (2) \quad [\textit{ Art. VIII-25 }]$$

At G where this normal intersects the x-axis, $y=0$.

Hence, CG = the value of x obtained by putting $y=0$ in (2).

$$= x_1 - \frac{b^2 x_1}{a^2}$$

$$= \frac{a^2-b^2}{a^2}x_1 = e^2 x_1, \text{ since } b^2 = a^2(1-e^2).$$

Hence, $SG = SC + CG = ae + e^2 x_1 = e(a+ex_1)$

and $S'G = CS' - CG = ae - e^2 x_1 = e(a-ex_1)$.

$\therefore \quad \dfrac{SG}{S'G}=\dfrac{a+ex_1}{a-ex_1}=\dfrac{SP}{S'P}$ *from* (1)

i.e., G divides the base SS' of the triangle SPS' internally in the ratio of the sides SP and $S'P$.

Hence, PG bisects the angle SPS'.

Cor. Since the tangent at P is perpendicular to the normal, it follows that the tangent bisects the angle SPS' externally. Hence, from the proposition proved above, it also follows that :

The tangent at any point of an ellipse makes equal angles with the focal distances of the point.

5. *If SY and $S'Y'$ be drawn perpendiculars from the foci of an ellipse upon the tangent at any point of it, then*

(*i*) $SY.S'Y'=b^2$

and (*ii*) Y *and* Y' *lie on the auxiliary circle.*

(*See figure of Prop. 4*).

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(*i*) The equation of any tangent YPY' to the ellipse is

$$y=mx+\sqrt{a^2m^2+b^2}$$

i.e.,
$$y-mx-\sqrt{a^2m^2+b^2}=0. \qquad \cdots \quad (1)$$

The points S and S' are respectively $(-ae, 0)$ and $(ae, 0)$.

Hence, $$SY=\frac{mae-\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}$$

and $$S'Y'=\frac{-mae-\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}.$$

$$\therefore \quad SY.S'Y'=\frac{(a^2m^2+b^2)-m^2a^2e^2}{1+m^2}$$

$$=\frac{m^2a^2(1-e^2)+b^2}{1+m^2}$$

$$=\frac{m^2b^2+b^2}{1+m^2}$$

$$=b^2.$$

(*ii*) Let SY and $S'P$ be produced to meet at the point K.

Since, the tangent PY bisects the angle SPS' externally (*Prop. 4. Cor.*), we have

$$\angle SPY=\angle KPY.$$

Hence, the triangles SPY and KPY are congruent.

$\therefore \qquad SP=PK$ and $SY=YK$.

Now Y being the middle point of SK and C being the middle point of SS',

$$\begin{aligned} CY &= \tfrac{1}{2}\, S'K \\ &= \tfrac{1}{2}\,(S'P+PK) \\ &= \tfrac{1}{2}\,(S'P+SP) \\ &= \tfrac{1}{2}.\, 2a = a. \end{aligned}$$

$\therefore$ Y lies on the circle described on the major axis as diameter, *i.e.*, on the auxiliary circle. Similarly, it can be proved that Y' also lies on the same circle.

Alternative method.

Any tangent to the ellipse is

$$y=mx+\sqrt{a^2m^2+b^2} \qquad \cdots \qquad \cdots \quad (1)$$

Also the equation to the line through the focus $(-ae,\ 0)$ perpendicular to (1) is

$$y=-\frac{1}{m}\,(x+ae) \qquad \cdots \qquad \cdots \quad (2)$$

If $(h,\ k)$ be the point of intersection of (1) and (2), the coordinates satisfy the equations. Hence,

$$k=mh+\sqrt{a^2m^2+b^2}$$

and $$k=-\frac{1}{m}(h+ae)$$

whence, $\quad (k-mh)^2+(mk+h)^2=a^2m^2+b^2+a^2e^2,$

i.e., $$\begin{aligned}(h^2+k^2)(1+m^2) &= a^2m^2+b^2+a^2-b^2 \\ &= a^2(1+m^2),\end{aligned}$$

i.e., $h^2+k^2=a^2$, a result independent of m and hence true for all positions of the tangent.

Hence, $(h,\ k)$ lies on the locus

$$x^2+y^2=a^2$$

which is clearly the auxiliary circle.

WORKED OUT EXAMPLES

Ex. 1. *Find the equations of the tangents to the ellipse $2x^2+3y^2=1$, which are parallel to the line $2x-y+3=0$.*

The given line is $y=2x+3$.

Any line parallel to it is $y=2x+c$, ... (1)

so that m, the gradient of the line is 2.

The equation to the ellipse can be written as

$$\frac{x^2}{\frac{1}{2}}+\frac{y^2}{\frac{1}{3}}=1, \quad \ldots \quad \ldots \quad (2)$$

so that $a^2=\frac{1}{2}$ and $b^2=\frac{1}{3}$.

If the line (1) is a tangent to the ellipse (2), we must have

$$c^2=a^2m^2+b^2$$
$$=\tfrac{1}{2}.4+\tfrac{1}{3}$$
$$=\tfrac{7}{3}.$$

$\therefore$ $$c=\pm\sqrt{\tfrac{7}{3}}.$$

The required equations of the tangents are therefore

$$y=2x\pm\sqrt{\tfrac{7}{3}}.$$

Otherwise: The line $y=2x+c$... (1) which represents any line parallel to the given line, meets the ellipse $2x^2+3y^2=1$... (2) in points whose abscissæ are given by

$$2x^2+3(2x+c)^2=1,$$

i.e., by $$14x^2+12cx+3c^2-1=0. \quad \ldots \quad \ldots \quad (3)$$

If the line (1) is a tangent, the two points of intersection coincide and hence the roots of equation (3) must be equal, so that $$(12c)^2-4.14\,(3c^2-1)=0,$$

i.e., $$144c^2-168c^2+56=0,$$

i.e., $$24c^2=56,$$

i.e., $$c=\pm\sqrt{\tfrac{7}{3}}.$$

Hence, the required equations to the tangents are

$$y=2x\pm\sqrt{\tfrac{7}{3}}.$$

Ex. 2. *If any tangent to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *intercepts lengths* h *and* k *on the axes, prove that* $\frac{a^2}{h^2}+\frac{b^2}{k^2}=1.$ [*C. U.*]

The equation of any tangent to the given ellipse is

$$y=mx+\sqrt{a^2m^2+b^2}. \quad \cdots \quad \cdots \quad (1)$$

It meets the axis of x where $y=0$.

$\therefore$ h = value of x obtained by putting $y=0$ in (1)

$$=-\frac{\sqrt{a^2m^2+b^2}}{m}. \quad \cdots \quad \cdots \quad (2)$$

Similarly, k = value of y obtained by putting $x=0$ in (1)

$$=\sqrt{a^2m^2+b^2}. \quad \cdots \quad \cdots \quad (3)$$

From (2) and (3), $\frac{a^2}{h^2}=\frac{a^2m^2}{a^2m^2+b^2}$ and $\frac{b^2}{k^2}=\frac{b^2}{a^2m^2+b^2}$

whence adding $\frac{a^2}{h^2}+\frac{b^2}{k^2}=1.$

Ex. 3. *Find the condition that the line* $lx+my=n$ *may be a normal to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

If the line $$lx+my=n \quad \cdots \quad \cdots \quad (1)$$

be a normal, let it be normal at the point (x_1, y_1) of the ellipse. It must then be identical with [*Art. VIII-25*]

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}$$

i.e., with $$\frac{a^2}{x_1}\cdot x-\frac{b^2}{y_1}\cdot y=a^2-b^2 \quad \cdots \quad \cdots \quad (2)$$

Comparing equations (1) and (2), we get

$$\frac{l}{a^2/x_1}=\frac{m}{-b^2/y_1}=\frac{n}{a^2-b^2}$$

i.e., $$\frac{x_1l}{a^2}=\frac{y_1m}{-b^2}=\frac{n}{a^2-b^2}.$$

$\therefore$ $$\frac{x_1}{a}=\frac{an}{l(a^2-b^2)} \text{ and } \frac{y_1}{b}=-\frac{bn}{m(a^2-b^2)}.$$

Answers :

1. (*i*) same ; (*ii*) same ; (*iii*) opposite ; (*iv*) opposite ; (*v*) same.
2. (*i*) opposite ; (*ii*) same ; (*iii*) same ; (*iv*) opposite ; (*v*) same.

4. (*i*) 1 ; (*ii*) $\frac{4}{13}$; (*i* 3 ; (*iv*) $\frac{3}{\sqrt{5}}$; (*v*) $\sqrt{3}-\frac{1}{2}$.

5. $\sqrt{h^2+k^2}$. 7. $\left\{\frac{a}{b}(b\pm\sqrt{a^2+b^2}), 0\right\}$. 8. $y=a$, $4x-3y+3a=0$.

9. $(3l-2m+n)^2=25(l^2+m^2)$. 10. (*i*) 1 ; (*ii*) $4\frac{1}{3}$.

11. (*i*) $4x-22y+15=0$ and $11x+2y+20=0$, $4x-22y+15=0$;

(*ii*) $x-y=0$ and $x+y-2=0$, $x-y=0$;

(*iii*) $11x-3y+3=0$ and $27x+99y+31=0$, $11x-3y+3=0$.

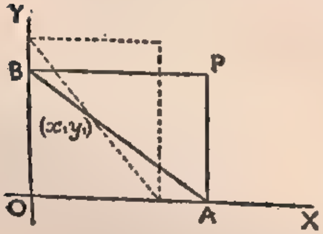
12. $B(x-h)-A(y-k)=\pm(Ax+By+C)$.

14. $x=y$, $x+3y-4=0$, $2x+y-3=0$.

IV-19. Locus Problems :

Problem 1. *A variable line passes through a fixed point (x_1, y_1) and meets the axes in A and B. If the rectangle $OAPB$ is completed, find the locus of P.*

Let the coordinates of P any point on the locus be (h, k).



Then $OA=h$ and $OB=AP=k$

The equation to the line AB is

then $$\frac{x}{h}+\frac{y}{k}=1.$$

For all positions of P, this line must pass through the fixed point (x_1, y_1).

Hence, $$\frac{x_1}{h}+\frac{y_1}{k}=1.$$

$\therefore$ (h, k) satisfies the equation

$$\frac{x_1}{x}+\frac{y_1}{y}=1$$

which is therefore the required equation to the locus.

Problem 2. *A line moves so that the sum of the intercepts made by it on the axes is always constant. Find the locus of the middle point of the portion of the line intercepted between the axes.*

Let AB be one position of the variable line and $P(h, k)$ its middle point. Clearly $OA=2h$ and $OB=2k$. From the given condition, we have

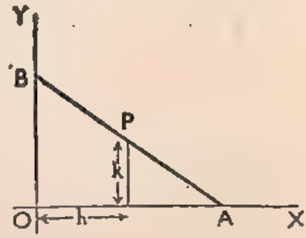
$$2h+2k=\text{constant}=2l \text{ (say)}$$

i.e., $h+k=l$

$\therefore$ (h, k) lies on the locus given by

$$x+y=l$$

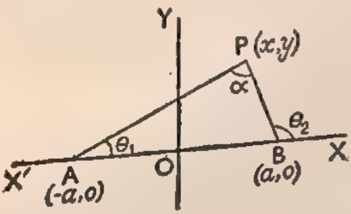
which is a straight line.



Problem 3. *The base of a triangle is fixed. Find the locus of the vertex, when it moves so that the vertical angle is always constant.*

Let AB be the fixed base of length $2a$ and let α be the constant vertical angle.

We choose O, the middle point of AB as origin, OB the axis of x and a line through O perpendicular to AB as the axis of y.



We have clearly $OA=OB=a$, so that the points A and B are respectively $(-a, 0)$ and $(a, 0)$

Let $P(x, y)$ be any point on the locus. If then θ_1 and θ_2 are the angles which AP and BP make with the x-axis, we have

$$\alpha=\theta_2-\theta_1$$

$$\therefore \quad \tan\alpha=\tan(\theta_2-\theta_1)=\frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_2\tan\theta_1}$$

where $\quad \tan\theta_1=\text{gradient of } AP=\dfrac{y}{x+a}$

and $\quad \tan\theta_2=\text{gradient of } BP=\dfrac{y}{x-a}.$

Hence $$\tan\alpha=\frac{\dfrac{y}{x-a}-\dfrac{y}{x+a}}{1+\dfrac{y}{x-a}\cdot\dfrac{y}{x+a}}=\frac{2ay}{x^2+y^2-a^2},$$

9. *Find the points on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at which the tangents are equally inclined to the axes. Find also the area of the square formed by the tangents at these points.

10. Find the equations of the common tangents to the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

11. Find the equation of each of the two tangents to the ellipse

$$\frac{x^2}{25}+\frac{y^2}{9}-1=0$$

from the point $(-15, -7)$. [*C. U.*]

12. Find the length of the chord intercepted

(*i*) by the ellipse $3x^2+5y^2=32$ on the line $x+y=4$.

(*ii*) by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the straight line $y=mx+c$.

13. Show that the diameters whose equations are $x-3y=0$ and $2x+y=0$ are conjugate diameters of the ellipse $2x^2+3y^2=6$.

14. Show that $4x-3y+4=0$ and $x+3y-7=0$ are parallel to the conjugate diameters of the ellipse $4x^2+9y^2=36$. [*C. U.*]

15. For the ellipse $8x^2+12y^2=96$ find a pair of conjugate semi-diameters including an angle $\tan^{-1} 7$. [*C. U.*]

16. Calculate the eccentric angles of the two points P (8, 3) and Q (6, −4) on the ellipse $\frac{x^2}{100}+\frac{y^2}{25}=1$ and satisfy yourselves that the semi-diameters with P, Q as end-points are conjugate to each other. [*C. U.*]

17. Show that the condition that (x_1, y_1) and (x_2, y_2) may be the extremities of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=0$; and show that in this case $x_1y_2-x_2y_1=\pm ab$. [*C. U.*]

[**Hints :** For the second part, multiply the relations $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ and $\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1$ and subtract from the result, $\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}\right)^2=0$.]

18. **P and Q are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angles are ϕ and ϕ'. If CP and CQ be a pair of conjugate diameters of the ellipse, prove that**

(*i*) $\tan\phi . \tan\phi' = -1$;

(*ii*) $\phi - \phi' = 90°$;

(*iii*) $CP^2+CQ^2=a^2+b^2$.

19. In the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ construct the equation to that particular chord which is bisected at the point $(2, -1)$. [C. U. 1956]

20. Find the locus of the middle points of chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which pass through a fixed point (h, k).

21. Find the locus of the point of intersection of a pair of tangents of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(i) when the difference of their gradients is a constant quantity $2k$;

(ii) which include a given angle α.

22. Tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the points where it is intersected by the straight line $lx+my+n=0$. Find the condition that these tangents may be at right angles.

[**Hints** : If (h, k) be the point where the tangents intersect, then $\frac{xh}{a^2}+\frac{yk}{b^2}=1$ and $lx+my+n=0$ must be identical. Now compare coefficients and apply the condition that (h, k) lines on the director circle.]

23. Show that the locus of the point of intersection of tangents at two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ when the difference of their eccentric angles is $2k$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2 k$. [C. U.]

[**Hints** : Take the eccentric angles as $\phi+k$ and $\phi-k$.]

24. Find the locus of the point of intersection of tangents at the extremities of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

[**Hints** : The eccentric angles of the extremities of a pair of conjugate diameters may be taken as ϕ and $90°+\phi$ [*Ex. 18 (ii)*]. Find the tangents at these points and eliminate ϕ.]

25. An ellipse slides between two straight lines mutually at right angles. Show that the locus of its centre is a circle. [C. U.]

[Since, it is known that the locus of the point of intersection P of two perpendicular tangents to the ellipse is the director circle, the distance of P from the centre of the ellipse is constant and equal to the radius of this circle. If now a pair of perpendicular tangents meeting at O be kept fixed and the ellipse be made to slide between them, the centre of the ellipse in all its positions will maintain the same distance from O and hence will move on a circle with centre O and radius equal to the radius of the director circle.]

26. Prove that the tangent at any point on an ellipse and the tangent at the corresponding point of the auxiliary circle intersect on the major axis.

27. **Prove that the product of the focal distances of any point P on an ellipse is equal to the square on the semi-diameter conjugate to CP.**

[**Hints** : If ϕ be the eccentric angle of P, then the focal distances are $a+ae\cos\phi$ and $a-ae\cos\phi$ (*Art. VIII-12*) ; also the eccentric angle of the extremity of the conjugate diameter is $\phi+90°$.]

Answers :

1. $x+2y=8$; $2x-y=1$. 4. $(-2, 1), (2, -1)$.

5. $x-3y+2=0, (-1, \frac{1}{3})$; $x-3y-2=0, (1, -\frac{1}{3})$.

9. $\left(\pm\dfrac{a^2}{\sqrt{a^2+b^2}}, \pm\dfrac{b^2}{\sqrt{a^2+b^2}}\right)$; $2(a^2+b^2)$.

10. $y=\pm x\pm\sqrt{a^2+b^2}$. 11. $x-4y-13=0$ and $4x-5y+25=0$.

12. (i) $\sqrt{2}$; (ii) $\dfrac{2ab}{a^2m^2+b^2}\sqrt{(1+m^2)(a^2m^2+b^2-c^2)}$.

15. $2x-y=0$, $x+3y=0$ and $2x+y=0$, $x-3y=0$.

16. $\tan^{-1}\frac{3}{4}$ and $\tan^{-1}(-\frac{4}{3})$. 19. $8x-9y=25$.

20. $b^2x^2+a^2y^2-b^2hx-a^2ky=0$.

21. (i) $k^2(x^2-a^2)^2-(b^2x^2+a^2y^2-a^2b^2)=0$.
(ii) $(x^2+y^2-a^2-b^2)^2\tan^2\alpha=4(b^2x^2+a^2y^2-a^2b^2)$.

22. $a^4l^2+b^4m^2=n^2(a^2+b^2)$. 24. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2$.

CHAPTER IX

THE HYPERBOLA

IX-1. Definitions :

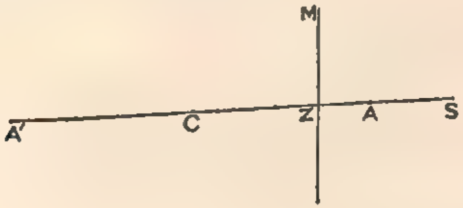
A hyperbola is a conic section of which the eccentricity e is greater than unity. We can therefore define it as follows :

A **Hyperbola** is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point in the plane to its distance from a fixed straight line in the same plane is a constant quantity greater than unity.

The fixed point is called the **Focus**, the fixed straight line is called the **Directrix**, and the constant ratio is called the **Eccentricity** denoted by the letter e.

IX-2. To prove that for a hyperbola

$$\mathbf{CS=ae \quad and \quad CZ=\frac{a}{e}}$$



Let S be the focus, MZ the directrix and e the eccentricity.

Draw SZ perpendicular to the directrix.

Since $e>1$, there will be a point A on SZ and another point A' on SZ produced, such that

$$\frac{SA}{AZ}=\frac{SA'}{A'Z}=e,$$

i.e., $\quad SA=e.AZ\cdots$ (1) and $SA'=e\cdot A'Z\cdots$ (2)

Hence, from the definition A and A' are points on the hyperbola.

Let the length AA' be called $2a$ and let C be the middle point of AA', so that $CA=CA'=a$.

From (1) and (2), we have

$$SA+SA'=e\,(AZ+A'Z),$$

or, $(CS-CA)+(CS+CA')=e.AA'=2ae.$

$\therefore$ $\mathbf{CS=ae}$ ... ... (i)

Again, $SA'-SA=e\,(A'Z-AZ),$

or, $AA'=e[(CA'+CZ)-(CA-CZ)],$

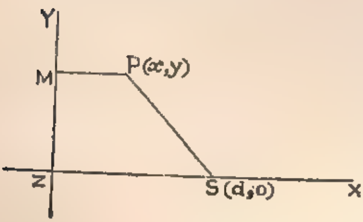
i.e., $2a=2e.CZ,$

$\therefore$ $\mathbf{CZ=\frac{a}{e}}$ ... ... (ii)

IX-3. Equation of the hyperbola.

To find the equation to a hyperbola referred to the directrix and perpendicular from the focus upon the directrix as axes of coordinates.

Let S be the focus, ZM the directrix and e the eccentricity.



SZ is drawn perpendicular to the directrix and ZS is produced to X. Let ZX, ZM be taken as the axes of coordinates.

As in the case of the ellipse let S be (d, o), so that if $P(x, y)$ be any point on the locus, the condition

$$SP^2=e^2.PM^2$$

gives $$(x-d)^2+y^2=e^2x^2.$$

Since in this case $e>1$, the equation is written as

$$\mathbf{x^2(e^2-1)-y^3+2dx-d^2=0.}$$

This, then, is the required equation.

IX-4. The standard equation :

We have $d=ZS$

$$=CS-CZ \text{ (Ref. Fig. Art. IX-2)}$$

$$=ae-\frac{a}{e}$$

$$=\frac{a}{e}\,(e^2-1).$$

$\therefore$ Substituting for d in the equation derived in the last article, we get

$$x^2(e^2-1)-y^2+2\frac{a}{e}(e^2-1)x-\frac{a^2}{e^2}(e^2-1)^2=0$$

i.e., $$(e^2-1)\left\{x^2+2\frac{a}{e}x\right\}-y^2=\frac{a^2}{e^2}(e^2-1)^2$$

i.e., $$(e^2-1)\left\{x^2+2\frac{a}{e}x+\frac{a^2}{e^2}\right\}-y^2=\frac{a^2}{e^2}(e^2-1)^2+\frac{a^2}{e^2}(e^2-1)$$

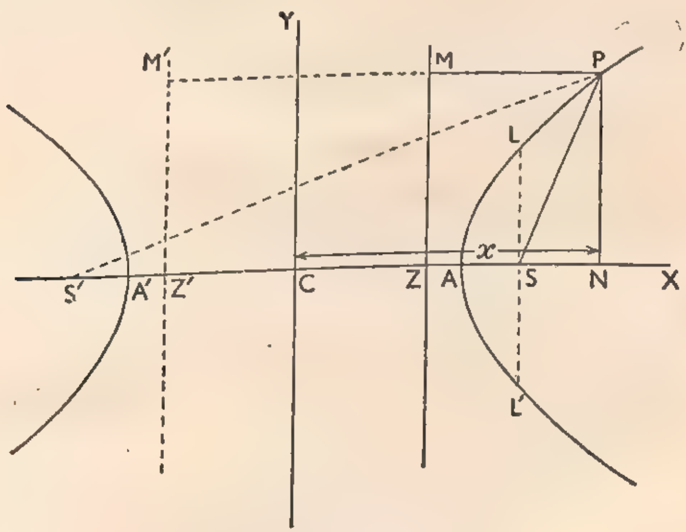
which reduces to

$$(e^2-1)\left(x+\frac{a}{e}\right)^2-y^2=a^2(e^2-1)$$

whence, $$\frac{\left(x+\frac{a}{e}\right)^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1.$$

Let us take the new origin at $C\left(-\frac{a}{e}, 0\right)$ $\left[\because CZ=\frac{a}{e}\right]$

Then the transformed equation is $\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1.$



Since $e>1$, the quantity $a^2(e^2-1)$ is positive.

If we put $a^2(e^2-1)=b^2$, the equation reduces to

$$\frac{\mathbf{x^2}}{\mathbf{a^2}}-\frac{\mathbf{y^2}}{\mathbf{b^2}}=\mathbf{1}$$

which is the **standard equation** of the curve.

We may, however, derive the standard equation independently thus :

With the same construction as in *Art. IX-2* we have

$$CS = ae \text{ and } CZ = \frac{a}{e}$$

We choose C as the origin, CS as the axis of x, and CY a line through C perpendicular to AA' as the axis of y.

Let $P(x, y)$ be any point on the curve.

Draw the ordinate PN and also draw PM perpendicular to the directrix.

Now, the condition satisfied by P is

$$SP = e.PM.$$

$\therefore$ $$SP^2 = e^2PM^2 = e^2ZN^2,$$

i.e., $$(x-ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2,$$

i.e., $$x^2(e^2-1) - y^2 = a^2(e^2-1)'$$

whence, $$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad \ldots \quad \ldots \quad (3)$$

i.e., $$\frac{\mathbf{x}^2}{\mathbf{a}^2} - \frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1}, \text{ on putting } b^2 = a^2(e^2-1).$$

IX-5. Second focus and second directrix :

In the figure of the previous article, let S' and Z' be points on the negative side of the x-axis such that

$$CS' = CS = ae$$

and $$CZ' = CZ = \frac{a}{e}.$$

Draw $Z'M'$ perpendicular to CZ' and PM' perpendicular to $Z'M'$ and join PS'.

The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1,$$

or, $$x^2(e^2-1) - y^2 = a^2(e^2-1),$$

i.e., $$x^2 + a^2e^2 + y^2 = a^2 + e^2x^2$$

which can be written as

$$(x+ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2,$$

i.e., $(CN+CS')^2+PN^2=e^2(CN+CZ')^2$,

i.e., $S'N^2+PN^2=e^2Z'N^2$,

i.e., $S'P^2=e^2PM'^2$

i.e., the distance of P from $S'=e\times$the distance of P from $Z'M'$.

Hence, the same curve might have been described with S' as focus and $Z'M'$ as directrix. In other words the hyperbola has a second focus and a second directrix.

IX-6. Definitions :

Vertices. The points A and A' where the line joining the foci meets the curve are called the vertices of the hyperbola.

Centre. The middle point of AA', *i.e.*, C is called the centre of the hyperbola.

Axes. The line AA' is called the **Transverse axis** of the hyperbola. If two points B and B' are taken on the y-axis such that $CB=CB'=b$, then the line BB' is called the **Conjugate axis** of the hyperbola.

Latus rectum. The double ordinate passing through the focus is called the latus rectum of the hyperbola.

Remark : If we put $x=0$ in the equation of the hyperbola we get $y^2=-b^2$, so that y is imaginary. This shows that the curve does not meet the axis of y in real points. The points B and B' whose coordinates are respectively $(0, b)$ and $(0, -b)$ are clearly not points on the curve.

IX-7. Geometrical property expressed by the equation :

The equation of the hyperbola can be written as

$$\frac{y^2}{b^2}=\frac{x^2}{a^2}-1,$$

or,
$$\frac{y^2}{b^2}=\frac{x^2-a^2}{a^2},$$

i.e.,
$$\frac{y^2}{(x+a)(x-a)}=\frac{b^2}{a^2},$$

i.e.,
$$\frac{PN^2}{AN.NA'}=\frac{CB^2}{CA^2},$$ which may be stated as :

The square on the ordinate of any point on a hyperbola varies as the rectangle contained by the segments of the transverse axis made by the ordinate.

IX-8. Shape of the curve :

Consider the standard equation of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The equation can be written as

$$\frac{y^2}{b^2}=\frac{x^2}{a^2}-1, \text{ or, } y=\pm\frac{b}{a}\sqrt{x^2-a^2}.$$

It follows that—

If $x^2<a^2$, *i.e.*, $(x+a)(x-a)<0$, *i.e.*, x lies between $-a$ and $+a$, y is imaginary showing that there is no point of the curve between the lines $x=-a$ and $x=+a$, the curve lying to the left of the line $x=-a$ and to the right of $x=a$.

If $x^2=a^2$, *i.e.*, $x=a$, or, $-a$, we get two equal values of y namely zero. Hence, the lines $x=a$ and $x=-a$ are tangents at A and A' respectively.

If $x^2>a^2$, *i.e.*, $(x+a)(x-a)>0$, *i.e.*, $x>a$, or, $<-a$, we get two equal and opposite values of y, showing that the curve is symmetrical with respect to the axis of x.

Again, writing the equation in the form

$$\frac{x^2}{a^2}=1+\frac{y^2}{b^2}, \text{ i.e., } x=\pm\frac{a}{b}\sqrt{y^2+b^2},$$

it is seen that y can have any real value without any limitation, and for all values of y we get two equal and opposite values of x. The curve is therefore, symmetrical with respect to the axis of y.

Also as one coordinate increases numerically, the other also increases. The curve therefore consists of two branches each extending to infinity in two directions as shown in the figure of Art. IX-4.

IX-9. The eccentricity :

The relation

$$b^2=a^2(e^2-1)$$

gives

$$e^2=\frac{a^2+b^2}{a^2},$$

which gives the eccentricity when the lengths of the transverse and conjugate axes of the hyperbola are known.

It may be noted that the conjugate axis of a hyperbola is less or greater than the transverse axis,

according as $b^2 <$ or $> a^2$,

i.e., " " $\dfrac{b^2}{a^2} <$ or > 1,

i.e., " " $\dfrac{b^2+a^2}{a^2} <$ or > 2,

i.e., " " $e^2 <$ or > 2,

i.e., according as $e <$ or $> \sqrt{2}$.

IX-10. The latus rectum (LSL′) :

If SL be the ordinate corresponding to the focus S, then the coordinates of the point L are $(ae,\ SL)$.

Hence, $\dfrac{a^2e^2}{a^2} - \dfrac{SL^2}{b^2} = 1.$

$\therefore$ $SL^2 = b^2(e^2-1) = b^2 . \dfrac{b^2}{a^2}.$

$\therefore$ The semi-latus rectum $SL = \dfrac{b^2}{a}$.

Hence, the latus rectum $\mathbf{LSL' = \dfrac{2b^2}{a}}$.

IX-11. An important property :

To prove that the difference of the focal distances of any point on the hyperbola is constant and equal to the transverse axis.

If $P(x, y)$ be any point on the curve (*fig. Art. IX-4*).

$$SP = e.PM = e.ZN = e(CN - CZ) = e\left(x - \frac{a}{e}\right) = ex - a.$$

$$S'P = e.PM' = e\ Z'N = e(CN + Z'C) = e\left(x + \frac{a}{e}\right) = ex + a.$$

Hence, $S'P - SP = (ex+a) - (ex-a) = 2a$

$=$ the transverse axis AA'.

IX-12. Mechanical construction :

Take two strings of unequal length, one being longer than the other by $2a$ the transverse axis of the hyperbola. Let S and S' be the positions of the foci of the curve. Fasten one extremity of the longer string at S' and one extremity of the shorter string at S and hold the free ends together. The longer string $S'R$ is kept taut while the shorter one is kept in contact with it by means of a pencil placed at P, as in the figure. If now the string, $S'R$ is rotated about S' the point P of the pencil as it moves about on the paper will trace a curve which is one branch of a hyperbola, for,

$$S'P-SP=(S'P+PR)-(SP+PR)$$

= the difference between the lengths of the strings

= the transverse axis.

The other branch will be traced by fixing the longer string at S and the shorter one at S'.

IX-13 Results similar to those of the ellipse :

Since the standard equation of the hyperbola differs from that of the ellipse only in having $-b^2$ in place of b^2, many results for the hyperbola will be obtained by simply writing $-b^2$ for b^2 in the corresponding result for the ellipse. The method of derivation would be exactly the same as in the case of the ellipse. Thus,

(1) The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{\mathbf{xx_1}}{\mathbf{a^2}}-\frac{\mathbf{yy_1}}{\mathbf{b^2}}=\mathbf{1} \qquad [\textit{ Ref. Art. VIII-20 }]$$

(2) The condition that the straight line $y=mx+c$ should be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\mathbf{c=\pm\sqrt{a^2m^2-b^2}} \qquad [\textit{ Ref. Art. VIII-21 }]$$

and hence,

$$\mathbf{y=mx+\sqrt{a^2m^2-b^2}}$$

is always a tangent to the hyperbola.

(3) The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{-b^2}} \qquad \text{[\textit{Ref. Art. VIII-25}]}$$

(4) The equation of the chord of contact of tangents drawn from (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

is $$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \qquad \text{[\textit{Ref. Art. VIII-26}]}$$

(5) The locus of the middle points of a system of chords parallel to $y=mx$ is given by the equation

$$y=\frac{b^2}{a^2m}x \qquad \text{[\textit{Ref. Art. VIII-27}]}$$

which is a line passing through the centre and is called a **Diameter** of the hyperbola.

(6) The condition that two diameters $y=mx$ and $y=m'x$ should be conjugate with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$mm'=\frac{b^2}{a^2} \qquad \text{[\textit{Ref. Art. VIII-28}]}$$

(7) As in *Art. VIII-24*, it can be shown that in general two tangents can be drawn from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and these are

$$y-y_1=m_1(x-x_1)$$

and $$y-y_1=m_2(x-x_1)$$

where m_1 and m_2 are the roots of the equation

$$(x_1^2-a^2)m^2-2x_1y_1m+y_1^2+b^2=0.$$

(8) The locus of the point of intersection of a pair of tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which meet at right angles is given by

$$x^2+y^2=a^2-b^2 \qquad \text{[\textit{Ref. Art. VIII-30}]}$$

which is called the **Director Circle.**

It will be seen that it is

(*i*) a real circle if $a > b$,

(*ii*) a point-circle if $a=b$, in which case the centre of the curve is the only point from which the tangents drawn to the hyperbola are at right angles, and

(*iii*) an imaginary circle if $a < b$, so that in this case no two tangents at right angles can be drawn to the curve.

IX-14. Rectangular or Equilateral hyperbola :

If the lengths of the transverse and conjugate axes of a hyperbola are, equal, *i.e.*, if $a=b$ the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ reduces to

$$\mathbf{x^2-y^2=a^2}.$$

This particular kind of hyperbola is called an equilateral or rectangular hyperbola.

For such a hyperbola, the eccentricity is given by

$$e^2=\frac{a^2+b^2}{a^2} \text{ where } a=b,$$

i.e., $$e^2=2, \text{ or, } \mathbf{e}=\sqrt{2}.$$

IX-15. Asymptotes :

Def : *A straight line which meets a curve in two points at infinity but which is itself not altogether at infinity is called an* **asymptote** *of the curve.*

To find the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The straight line $y=mx+c$ ... ... (1)
meets the hyperbola in points whose abscissæ are given by

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1,$$

i.e., $$x^2(b^2-a^2m^2)-2a^2mcx-a^2(b^2+c^2)=0 \quad \cdots \quad (2)$$

If the line (1) is an asymptote both the roots of equation (2) must be infinite, the conditions for which are

$$b^2-a^2m^2=0 \text{ and } a^2mc=0.$$

Hence, $m=\pm\frac{b}{a}$ and $c=0$ $(\because\ a\neq 0)$.

Since, we get two values of m, we conclude that there are two asymptotes of the curve whose equations are

$$\mathbf{y}=\frac{\mathbf{b}}{\mathbf{a}}\mathbf{x} \quad \text{and} \quad \mathbf{y}=-\frac{\mathbf{b}}{\mathbf{a}}\mathbf{x}$$

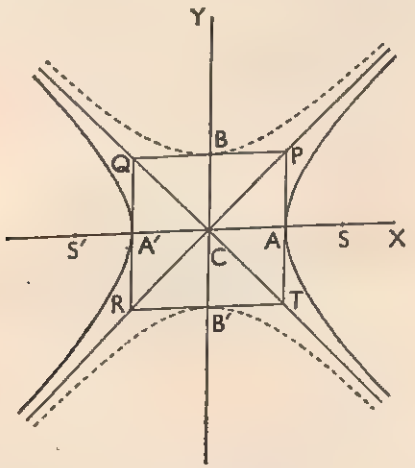
These lines are clearly equally inclined to the axis of x.

Cor. If $m=\pm\frac{b}{a}$, but $c\neq 0$, the equation (2) reduces to one in which the coefficient of x^2 is zero and hence, one root of the equation is infinite. Hence,

Any line parallel to an asymptote of the curve meets it in one finite point and in one point at infinity.

IX-16. Geometrical construction for the asymptotes :

Let AA' and BB' be the transverse and conjugate axes of the curve respectively, where $CA=CA'=a$ and $CB=CB'=b$.



Through A and A' draw lines parallel to the conjugate axis and through B and B' draw lines parallel to the transverse axis of the curve thus forming a rectangle $PQRT$. If now the diagonals PR and QT be joined and produced, these will clearly be the asymptotes of the hyperbola, for, it is easily seen that the lines pass through the centre C which is the origin and have gradients $\frac{b}{a}$ and $-\frac{b}{a}$, so that their equations are $y=\pm\frac{b}{a}x$.

IX-17. Asymptotes of the rectangular hyperbola :

For a rectangular hyperbola, we have $b=a$, so that the asymptotes of $x^2-y^2=a^2$ are given by the equations

$$y=x \quad \text{and} \quad y=-x$$

These are lines making angles 45° and -45° respectively with the axis of x and hence, are at right angles to one another.

IX-18. Conjugate hyperbola :

Def. : If AA' and BB' are respectively the transverse and conjugate axes of a hyperbola, then the hyperbola which has BB' for the transverse axis and AA' for the conjugate axis is called the **Conjugate hyperbola** of the first.

Consider the hyperbola given by the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \quad \cdots \quad \cdots \quad (1)$$

The hyperbola conjugate to (1) has transverse axis of length $2b$ along the axis of y and conjugate axis of length $2a$ along the axis of x and hence ts equation is

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1,$$

i.e.,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1. \quad \cdots \quad \cdots \quad (2)$$

The conjugate hyperbola is the dotted curve in the figure of *Art. IX-16.*

IX-19. Equation of a rectangular hyperbola referred to its asymptotes as axes :

Let OX' and OY' the asymptotes of the rectangular hyperbola be taken as axes of coordinates and let P be any point on the curve. Draw PN and PM perpendiculars to OX and OX' respectively and from M draw MR perpendicular to PN produced and MT perpendicular to OX.

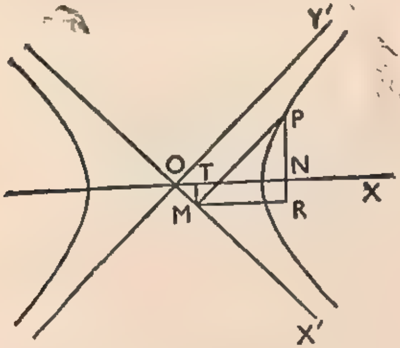
If the current P on the curve were taken as (X, Y) with reference to the transverse and conjugate axes as axes of reference, then the equation of the hyperbola would be

$$X^2-Y^2=a^2 \quad \cdots \quad \cdots \quad (1)$$

If, now, referred to OX' and OY' as axes, the point P be (x, y) so that $OM=x$ and $MP=y$, we must have

$$X=ON=OT+MR=OM\cos 45^\circ+MP\cos 45^\circ=\frac{x+y}{\sqrt{2}}$$

$$Y=NP=RP-MT=MP\sin 45^\circ-OM\sin 45^\circ=\frac{y-x}{\sqrt{2}}.$$



Hence, from (1), we get on substitution,

$$\left(\frac{x+y}{\sqrt{2}}\right)^2-\left(\frac{y-x}{\sqrt{2}}\right)^2=a^2,$$

or, $$\frac{4xy}{2}=a^2,$$

i.e., $$\mathbf{xy=\frac{a^2}{2}}$$

which is therefore the required equation of the hyperbola referred to its asymptotes as axes of coordinates.

IX-20. Some geometrical properties :

1. *The tangent at any point of a hyperbola bisects the angle between the focal distances of the point.*

Let $P(x_1, y_1)$ be a point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

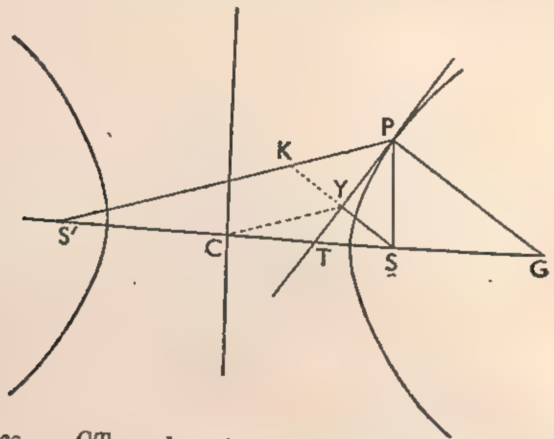
We have, $SP=ex_1-a$, $S'P=ex_1+a$. ... (1)

[*Art. IX-11*]

The equation of the tangent at P is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1. \qquad \ldots \qquad \therefore \qquad (2)$$

At T, where the tangent meets the x-axis, $y=0$.



Hence, $\quad CT=$ value of x obtained by putting $y=0$ in (2)

$$=\frac{a^2}{x_1}.$$

$\therefore \qquad S'T=S'C+CT=ae+\dfrac{a^2}{x_1}$

$\qquad ST=CS-CT=ae-\dfrac{a^2}{x_1}.$

Hence, $$\frac{S'T}{ST}=\frac{ae+\dfrac{a^2}{x_1}}{ae-\dfrac{a^2}{x_1}}=\frac{ex_1+a}{ex_1-a}=\frac{S'P}{SP} \qquad \textit{from } (1)$$

i.e., T divides the base $S'S$ of the triangle $S'PS$ internally in the ratio of the sides $S'P$ and SP.

Hence, PT bisects the angle $S'PS$.

Remark. : Since, the normal PG is perpendicular to the tangent PT, it follows that PG bisects the angle $S'PS$ externally, *i.e.*, *the normal at any point of a hyperbola bisects the angle between the focal distances of the point externally.*

Cor. It follows that an ellipse and a hyperbola having the same foci intersect at right angles.

2. *The foot of the perpendicular from the focus on any tangent to the hyperbola lies on the auxiliary circle.*

Def.: The circle described on the transverse axis AA' of a hyperbola as diameter is called the **Auxiliary circle.**

Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Any tangent to the hyperbola is

$$y=mx+\sqrt{a^2m^2-b^2}. \qquad \ldots \qquad \ldots \qquad (1)$$

Also the equation to the line through the focus $(ae, 0)$ perpendicular to (1) is

$$y=-\frac{1}{m}(x-ae). \qquad \ldots \qquad \ldots \qquad (2)$$

If (h, k) be the point of intersection of (1) and (2) the coordinates satisfy both the equations. Hence,

$$k=mh+\sqrt{a^2m^2-b^2}$$

and $$k=-\frac{1}{m}(h-ae)$$

whence, $$(k-mh)^2+(mk+h)^2=(a^2m^2-b^2)+a^2e^2,$$

i.e., $$(h^2+k^2)(1+m^2)=a^2m^2-b^2+a^2+b^2$$
$$=a^2(1+m^2),$$

i.e., $h^2+k^2=a^2$, a result independent of m and hence, true for all positions of the tangent.

Hence, (h, k) lies on the locus

$$x^2+y^2=a^2$$

which is clearly the auxiliary circle.

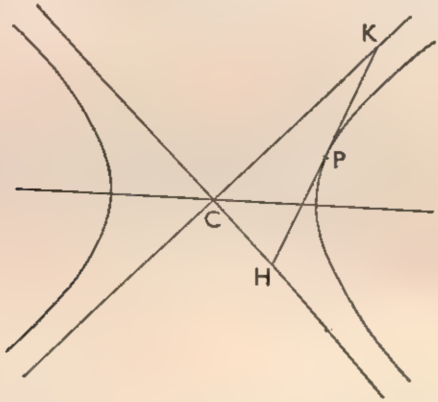
Alternative proof (by Geometry):

Produce the perpendicular SY to meet $S'P$ in K and join CY (*figure of Prop. 1*). Now prove as in the case of the ellipse that $CY=\frac{1}{2}S'K=\frac{1}{2}(S'P-SP)=a$.

3. *If the tangent at any point P of a hyperbola meets the directrix in Z, then the angle PSZ is a right angle.*

The proof is similar to that of the corresponding proposition (*Art. VIII-31. Prop. 3*) for the ellipse.

4. *The portion of the tangent at any point of a hyperbola intercepted by the asymptotes is bisected at the point of contact.*



Let the tangent at the point $P(\alpha, \beta)$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meet the asymptotes in the points $K(x_1, y_1)$ and $H(x_2, y_2)$.

The equation of the tangent at (α, β)

is $$\frac{x\alpha}{a^2}-\frac{y\beta}{b^2}=1 \qquad \cdots \qquad \cdots \quad (1)$$

The asymptotes are $$y=\frac{b}{a}x \qquad \cdots \qquad \cdots \quad (2)$$

and $$y=-\frac{b}{a}x \qquad \cdots \qquad \cdots \quad (3)$$

The coordinates of K will be obtained by solving (1) and (2).

$\therefore$ x_1 = abscissa of K

= value of x obtained from the equation

$$\frac{x\alpha}{a^2}-\frac{\beta}{b^2}\cdot\frac{b}{a}x=1,$$

i.e., $$\frac{x}{a}\left(\frac{\alpha}{a}-\frac{\beta}{b}\right)=1.$$

Hence, $$x_1=\frac{a}{\frac{\alpha}{a}-\frac{\beta}{b}}.$$

Again, solving (1) and (3), we get

$$x_2 = \text{abscissa of } H$$

$$= \frac{a}{\dfrac{\alpha}{a}+\dfrac{\beta}{b}}$$

$\therefore$ Abscissa of the middle point of HK

$$= \frac{x_1+x_2}{2} = \frac{1}{2}\left\{\frac{a}{\alpha/a-\beta/b}+\frac{a}{\alpha/a+\beta/b}\right\}$$

$$= \frac{a}{2}\cdot\frac{2\dfrac{\alpha}{a}}{\dfrac{\alpha^2}{a^2}-\dfrac{\beta^2}{b^2}} = \frac{\alpha}{\dfrac{\alpha^2}{a^2}-\dfrac{\beta^2}{b^2}}$$

$= \alpha$, since (α, β) is a point on the hyperbola.

Similarly, the ordinate of the middle point of HK will be found to be β.

Thus, the middle point of HK is (α, β) that is, the point P, which proves the proposition.

WORKED OUT EXAMPLES

Ex. 1. *Find the equation to the hyperbola whose foci are* $(-1, 3)$ *and* $(5, 3)$ *and whose eccentricity is* $\frac{3}{2}$.

The centre of the hyperbola is the middle point of the line joining the foci and so its coordinates are $(2, 3)$.

If a and b be the semi-transverse and semi-conjugate axes, then

$$2ae = \text{distance between the foci} = 6.$$

$\therefore$ $a = 2$, since $e = \frac{3}{2}$

and $b^2 = a^2(e^2-1)$ gives $b = \sqrt{5}$.

$\therefore$ The required equation is

$$\frac{(x-2)^2}{4} - \frac{(y-3)^2}{5} = 1,$$

i.e., $$5x^2 - 4y^2 - 20x + 24y - 36 = 0.$$

Ex. 2. *If e_1 and e_2 be the eccentricities of a hyperbola and its conjugate, show that*

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=1. \qquad [C. U.]$$

Let the equation to the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Then the equation to the conjugate hyperbola is

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1.$$

We have clearly,

$$e_1^2=\frac{a^2+b^2}{a^2} \text{ and } e_2^2=\frac{b^2+a^2}{b^2}.$$

$$\therefore \quad \frac{1}{e_1^2}+\frac{1}{e_2^2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}$$

$$=\frac{a^2+b^2}{a^2+b^2}=1.$$

Ex. 3. *If P be the foot of the perpendicular from the focus S of a hyperbola upon an asymptote, then prove that SP and CP are respectively equal to the semi-conjugate and semi-transverse axes.*

The equation to an asymptote is

$$y=\frac{b}{a}x, \qquad i.e., \quad bx-ay=0.$$

The perpendicular from $S(ae, 0)$ upon this line is

$$\frac{bae}{\sqrt{b^2+a^2}}=\frac{bae}{ae}=b.$$

$\therefore$ $\qquad SP=b$, the semi-conjugate axis.

Also, $\quad CP=\sqrt{CS^2-SP^2}=\sqrt{a^2e^2-b^2}=a$, the semi-transverse axis.

EXERCISE IX

1. Find the length of the axes, coordinates of the foci and the latus rectum of the hyperbola

$$25x^2-36y^2=225.$$

2. Calculate the eccentricity and the positions of the two foci of the hyperbola $\frac{x^2}{12^2}-\frac{y^2}{5^2}=1.$

[C. U. 1957]

3. In a hyperbola the distance between the foci is 10 and the conjugate axis is 6. Find its equation referred to its axes as axes of coordinates.

4. In a hyperbola the distance between the foci is double the distance between the vertices. If the conjugate axis is of length 6, find its equation referred to its axes as axes of coordinates.

5. Find the coordinates of the centre, the eccentricity and the coordinates of the foci of the hyperbola,

$$16x^2-9y^2+32x+36y-164=0.$$

6. Find the equation to the hyperbola whose foci are at (8, 3) and (0, 3) and eccentricity is $\frac{4}{3}$.

7. Find the equations of the tangents to the hyperbola $x^2-5y^2=40$ which are perpendicular to the straight line $2x-y+3=0$.

8. For the hyperbola $16x^2-9y^2=144$, find the equation to the diameter which is conjugate to the diameter whose equation is $x=2y$. [C. U.]

9. Find the eccentricity of the hyperbola which is conjugate to the hyperbola $16x^2-9y^2=72$.

10. Find the coordinates of the vertices and foci and the equations to the axes and directrices of the rectangular hyperbola $xy=\frac{a^2}{2}$.

11. Find the condition that $y=mx+c$ should touch the rectangular hyperbola $xy=\frac{a^2}{2}$.

12. Prove that if 2α be the angle between the asymptotes of a hyperbola, then $e=\sec\alpha$.

13. Ascertain the coordinates of the two points Q, R where the tangent to the hyperbola $\frac{x^2}{45}-\frac{y^2}{20}=1$, at the point P (9, 4) intersects the two asymptotes. Finally prove that P is the middle point of QR. [C. U. 1950]

14. Construct the equation of the hyperbola (π) whose transverse and conjugate axes are located respectively along the axes $y=0$ and $x=0$ and has a point-circle for its director circle, and besides passes through the point (13, −5). Finally compute the angle between the two asymptotes of π. [C. U. 1956]

15. Prove that the locus of the points of intersection of the straight lines $\frac{x}{a}+\frac{y}{b}=k$ and $\frac{x}{a}-\frac{y}{b}=\frac{1}{k}$ where k varies, is a hyperbola.

16. A circle always touches the axis of y and cuts off a chord of constant length from the axis of x. Prove that the locus of its centre is a rectangular hyperbola.

17. Find the locus of the middle points of the focal distances of points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

18. Prove that the locus of the mid-points of portions of tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, intercepted between the axes is $\frac{a^2}{x^2}-\frac{b^2}{y^2}=4$.

19. In a rectangular hyperbola, prove that

$$SP.S'P=CP^2.$$

20. The normal at any point P of a rectangular hyperbola meets the transverse and conjugate axes in G and G'. Prove that

$$PG=PG'=PC.$$

21. The ordinate through a point P of a hyperbola meets the asymptotes in Q and Q'. Prove that

$$PQ.PQ'=(\text{semi-conjugate axis})^2$$

22. Prove that the points of intersection of the asymptotes of a hyperbola with the directrices lie on the auxiliary circle.

23. Prove that the circle described on the line joining the foci of a hyperbola as diameter passes through the foci of the conjugate hyperbola.

24. Prove that the area of the triangle formed by the asymptotes and any tangent to a hyperbola is constant.

25. A series of rectangular hyperbolas have the same asymptotes. Show that if two lines form a pair of conjugate diameters with respect to one of them, they are so with respect to each one of them. [C. U.]

26. Find the equation of the tangent and the normal to the rectangular hyperbola $xy=c^2$ at the point $P\left(ct_1, \frac{c}{t_1}\right)$ and prove that if the normal at P meets the curve again at $Q\left(ct_2, \frac{c}{t_2}\right)$ then $t_1^3t_2=-1$.

Answers :

1. $6, 5$; $(\pm\frac{1}{2}\sqrt{61}, 0)$; $4\frac{1}{6}$.
2. $\frac{13}{12}$, $(\pm 13, 0)$.
3. $\frac{x^2}{16}-\frac{y^2}{9}=1$.
4. $3x^2-y^2=9$.
5. $(-1, 2)$; $\frac{5}{3}$; $(4, 2)$, $(-6, 2)$.
6. $7x^2-9y^2-56x+54y-32=0$,
7. $x+2y=\pm 2\sqrt{2}$.
8. $9y=32x$.
9. $\frac{5}{4}$.
10. vertices : $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$, $\left(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$;

    foci : (a, a), $(-a, -a)$; axes : $y=\pm x$; directrices : $x+y=\pm a$.
11. $c=\pm a\sqrt{-2m}$.
13. $(15, 10)$, $(3, -2)$.
14. $x^2-y^2=144$; $90°$.
16. If $2a$ be the length of the chord then the locus is $x^2-y^2=a^2$.
17. $\frac{(2x-ae)^2}{a^2}-\frac{4y^2}{b^2}=1$.
26. $\frac{x}{t_1}+yt_1=2c$; $t_1^2x-t_1y=ct_1^4-c$.

SOLID GEOMETRY

CHAPTER I

DEFINITIONS AND FIRST PRINCIPLES

1. In plane geometry we defined a plane surface as one which satisfies the condition that if two points taken at random on the surface be joined by a straight line then every point on this line must lie on the surface, and we dealt with properties of lines and figures that could be drawn on such a surface. Here, in Solid Geometry or Geometry of Three Dimensions we shall be dealing with points, lines, surfaces and solids lying in space.

LINES AND PLANES

2. Unless otherwise stated, we shall suppose that a straight line is of unlimited length and a plane is of infinite extent, from which the following conclusions are obvious.

(1) If a straight line is drawn on a plane, then when indefinitely produced both ways it lies wholly on that plane.

(2) By turning a plane about a straight line lying in it, it can be made to pass through any given point in space.

By 'a line' we shall always mean a straight line.

LINES IN SPACE

3. **Coplanar lines** : Two or more straight lines are said to be coplanar when they are drawn on a plane or when they are such that a plane may be made to pass through them.

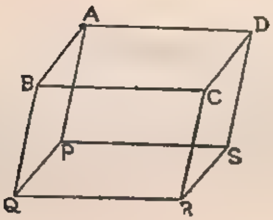
Thus two coplanar lines must either intersect or be parallel.

4. **Parallel lines** : Two straight lines are said to be parallel when they are coplanar and do not intersect though indefinitely produced.

5. **Skew lines** : Straight lines through which a plane

cannot be made to pass are said to be **Non-Coplanar** or Skew. Such lines are therefore neither parallel nor do they intersect.

Two straight lines may therefore exist in space in three distinct ways, *viz.*



(*i*) they may intersect, as PS, SR in the adjoining solid figure which represents a parallelopiped ;

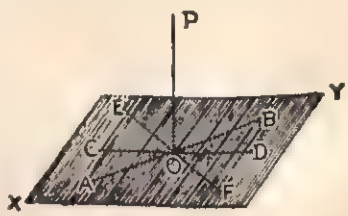
(*ii*) they may be parallel, as PS, QR ; or

(*iii*) they may be skew, as PS, QB.

A LINE AND A PLANE IN SPACE

6. **Line parallel to a plane** : A straight line and a plane are said to be parallel to one another when they do not meet though both are indefinitely produced.

7. **Line perpendicular to a plane** : A straight line is said to be perpendicular or **Normal** to a plane when it is perpendicular to every straight line drawn in the plane through the point where the line meets the plane.



Thus PO will be said to be perpendicular to the plane XY if it is perpendicular to the lines AOB, COD, EOF etc. all passing through O and lying in the plane XY.

It can be seen that a line and a plane may exist in space in **three** distinct ways :

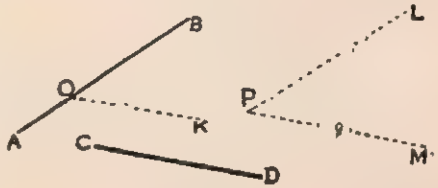
(*i*) the line may lie in the plane when they have an infinite number of points in common,

(*ii*) the line may intersect the plane when they have only one point in common, or

(*iii*) the line may be parallel to the plane when they have no point in common.

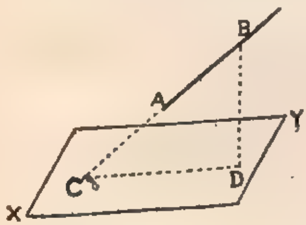
ANGLE BETWEEN TWO SKEW LINES, BETWEEN A LINE AND A PLANE AND BETWEEN TWO PLANES

8. **The angle between two skew lines** is measured by the angle between one of the lines and a line drawn through a point on it parallel to the other, or between two lines drawn through any point parallel to each of the given lines.



Thus in the figure the angle between the two skew lines AB, CD is measured either by the angle BOK or by the angle LPM, where OK and PM are each parallel to CD and PL is parallel to AB.

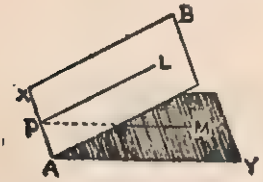
9. **The angle between a line and a plane** is measured by the angle which the line makes with another line drawn in the plane joining the point where the line (produced if necessary) meets the plane to the foot of the perpendicular drawn from any point of the line upon the plane.



Thus in the adjoining figure the angle which the straight line AB makes with the plane XY is measured by the angle BCD where C is the point in the plane XY where BA produced meets it and D is the foot of the perpendicular from B upon the plane XY.

10. **Angle between two planes : Dihedral angle :** When two planes intersect they are said to form a **dihedral angle.**

A dihedral angle is measured by the plane angle contained by two straight lines, one in each plane, drawn from a point on their line of section each perpendicular to this line.



Thus in the figure, if P be any point on the line of section AX of the planes AB and XY, and PL is drawn in the plane AB perpendicular to AX, and PM in the plane XY also perpendicular to AX, the angle LPM measures the dihedral angle formed by the two planes AB and XY.

Note : It is assumed that two planes intersect along a straight line.

PLANES IN SPACE

11. **Parallel planes :** Planes are said to be parallel when they do not meet though produced beyond limit in all directions.

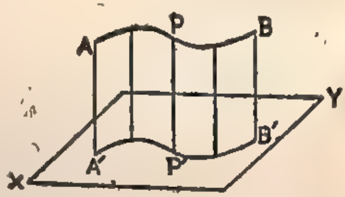
12. **Perpendicular planes :** Two planes are said to be perpendicular to one another when the angle which measures the dihedral angle formed by the planes is a right angle.

PROJECTIONS

13. **The projection of a point on a plane** is the foot of the perpendicular drawn from the point upon the plane.

14. **The projection of a line** (*straight or curved*) **on a plane** is the locus of the feet of the perpendiculars drawn from all points of the line upon the plane.

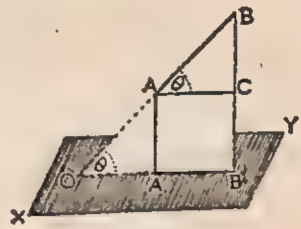
Thus in the figure, the projection of the line AB on the plane XY is the line $A'B'$ where AA'...PP'...BB' are the perpendiculars drawn from A...P...B on the plane XY.



It can be proved, as will be shown hereafter, that the projection of a straight line on a plane is also a straight line and that the line and its projection are coplanar. We can therefore give the following definition.

The projection of a straight line of limited length on a plane is the length intercepted between the feet of the perpendiculars drawn from the extremities of the line on the plane.

In the figure, the lines AA', BB' are drawn perpendiculars from A, B to the plane XY to meet it in A', B' respectively. Then $A'B'$ is the projection of AB on the plane XY.



It will now be seen that the angle which a straight line makes with a plane is measured by the angle between the line and its projection on the plane. For if AB and $A'B'$ be produced to meet in O, then the angle BOB' measures the angle which AB makes with the plane XY.

15. **Length of projection :** If AB makes an angle θ with the plane XY then clearly $\angle BOB'=\theta$.

From A draw AC parallel to $A'B'$ to meet BB' in C. Then

$$\angle BAC=\angle BOB'=\theta$$

$\therefore\quad A'B'=AC$, being opposite sides of a rectangle

$$=AB\cos BAC$$

i.e., $\mathbf{A'B'=AB\cos\theta}$.

Thus *the length of the projection of a finite straight line AB on a plane XY is equal to $AB\cos\theta$, where θ is the angle made by the straight line AB with the plane XY.*

AXIOMS

16. In discussing properties of lines and planes in space we shall base our deductions on the following fundamental truths which are regarded as axioms.

Axiom—1. One and only one plane passes through a given line and a given point outside it.

Axiom—2. If two planes have one point in common, they have at least a second point in common.

From the above the following conclusions also immediately follow :

(1) The position of a plane is fixed when it passes through
 (*a*) two intersecting straight lines ;
 (*b*) two parallel straight lines ;
 (*c*) three non-collinear points.

(2) The locus of a straight line which always
 (*a*) passes through a fixed point and intersects a fixed straight line ;
 (*b*) intersects two fixed intersecting straight lines ;
 (*c*) intersects two fixed parallel straight lines ;
 is a plane.

CHAPTER II

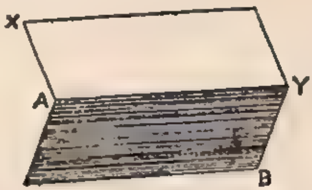
LINES AND PLANES IN SPACE

THEOREM 1

Two intersecting planes cut one another in a straight line and in no point outside it.

Let AB and XY be any two intersecting planes.

It is required to prove that the planes AB and XY intersect in a straight line and in no point outside it.



Since by hypothesis the planes intersect, they must have a point common to them. Let this point be A. Then they must have a second point also in common (Axiom 2). Let this point be Y. Then the straight line joining the common points A and Y must lie on both the planes ; in other words, the planes intersect along the straight line AY.

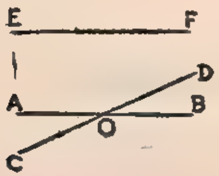
Also, if possible, let there be another point P outside the straight line AY which is common to both the planes. Then we have two distinct planes AB and XY passing through the line AY and the point P outside it, which is absurd. (Axiom 1)

Hence the two planes AB and XY intersect along the straight line AY, and in no point outside it.

Exercises

1. Prove that two intersecting straight lines cannot both be parallel to a third straight line.

[Let AB and CD intersect at O and let AB be parallel to EF.



AB and EF being parallel must be coplanar. If CD lies in this plane, then it cannot be parallel to EF, since it intersects AB, one of the two parallel straight lines AB and EF. If CD does not lie in the plane of AB, EF then, if possible, let CD and EF be coplanar. In that case the planes of AB, EF and of CD, EF have the line EF and the point O, outside EF common to them, which is absurd. Hence CD and EF cannot be coplanar and therefore cannot be parallel.]

2. Prove that a plane which contains only one of two given parallel straight lines is parallel to the other.

3. Two planes are drawn through two given parallel straight lines, one through each. Prove that their line of intersection is also parallel to the given lines.

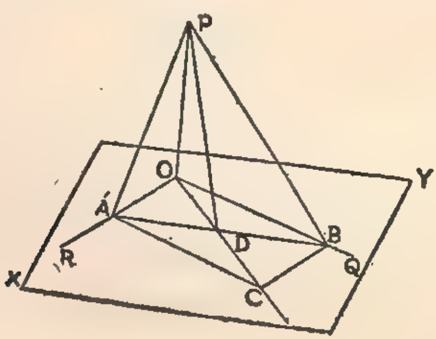
4. Two parallel planes are intersected by a third plane. Prove that the two lines of section are also parallel.

5. If two planes are such that the same straight line is perpendicular to both, then prove that the planes must be parallel.

6. Prove that the lines of section of three non-collinear planes are either concurrent or parallel.

THEOREM 2

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane in which they lie.



Let OP be perpendicular to each of the two intersecting straight lines OQ and OR at their point of intersection O and let XY be the plane in which OQ and OR lie.

It is required to prove that OP is perpendicular to the plane XY.

Through O draw *any* line OC in the plane XY and draw CA and CB parallel to OQ and OR respectively. Let the diagonals of the parallelogram formed intersect in D, so that D is the middle point of AB.

Join PA, PB and PD.

Since D is the middle point of the side AB of the triangle OAB, we have

$$OA^2+OB^2=2OD^2+2AD^2 \qquad \ldots \quad (1)$$

Similarly from the triangle PAB,

$$PA^2+PB^2=2PD^2+2AD^2 \qquad \ldots \quad (2)$$

Subtracting (1) from (2), we get

$$(PA^2-OA^2)+(PB^2-OB^2)=2(PD^2-OD^2) \qquad \ldots \quad (3)$$

Since PO is perpendicular to OA, OB

$$PA^2-OA^2=PO^2 \text{ and } PB^2-OB^2=PO^2.$$

$\therefore$ From (3), $2PO^2=2(PD^2-OD^2)$

or, $PO^2+OD^2=PD^2$.

Hence PO is perpendicular to *any* line OD which meets it in the plane XY.

$\therefore$ PO is perpendicular to the plane XY.

Exercises

1. OX, OY and OZ are three mutually perpendicular lines. OP is drawn perpendicular to XY. Prove that XY is perpendicular to the plane of the triangle OPZ.

2. OA, OB, OC are three straight lines which are mutually perpendicular. If AD is drawn perpendicular to BC, shew that OD is perpendicular to BC. [C. U. 1952]

3. If AB is perpendicular to a plane and if from B, the foot of the perpendicular, the line BE is drawn perpendicular to a line CE in the plane, show that CE is perpendicular to the plane of AE, BE. [C. U. 1950]

4. A straight line PQ is drawn in the plane XY and from a point A outside the plane, AB and AC are drawn perpendiculars to the plane XY and to the straight line PQ respectively. Prove that PQ is perpendicular to the plane of AB, AC.

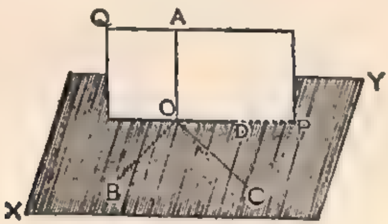
5. Prove that one and only one perpendicular can be drawn to a plane (*i*) at a given point of it, (*ii*) from a given point outside it.

6. Prove that there cannot be more than three mutually perpendicular straight lines in space meeting at a point. [*C. U.* 1948]

THEOREM 3

All straight lines drawn perpendicular to a given straight line at a given point on it are coplanar.

Let each of the straight lines OB, OC, OD be perpendiculars to the straight line OA at the point O.



It is required to prove that OB, OC, OD are coplanar. Let XY be the plane which passes through OB, OC and PQ the plane which passes through OA, OD and let OP be the common line of section of the planes XY and PQ.

Since OA is perpendicular to OB and OC, it is perpendicular to the plane XY in which OB, OC lie.

$\therefore$ OA is perpendicular to OP which meets it in the plane XY, *i.e.*, OP is perpendicular to OA.

Also, by hypothesis, OD is perpendicular to OA.

$\therefore$ OD and OP both lying in the plane PQ and being perpendicular to OA which is also a line in the same plane, must coincide.

Hence OD lies in the plane XY, *i.e.*, OB, OC, OD are coplanar.

Similarly, any other straight line drawn perpendicular to OA at O lies in the plane XY.

$\therefore$ All lines drawn perpendicular to OA at O are coplanar.

Exercises

1. Find the locus of a point in space equidistant from two given points. [C. U.]

[If A and B be the given points and P a point on the locus then the line PC joining P to the middle point C of AB must be perpendicular to AB. $\therefore$ Such points lie on lines through C perpendicular to AB. These lines all lie in a plane. Hence the locus is the plane which passes through the middle point of the line joining the given points and is perpendicular to this line.]

2. Find the locus of a point in space which is equidistant from the vertices of a triangle.

3. Prove that a point can be found in a plane equidistant from three points outside the plane. State the exceptional cases, if any. [C. U.]

4. Prove that all straight lines drawn perpendicular from a given point to a system of parallel straight lines in space are coplanar. [C. U.]

[**Hint** : Draw a straight line through the given point parallel to the given system and prove that all such lines drawn are perpendicular to this line.]

5. If a triangle revolves about its base, show that the vertex describes a circle. [C. U.]

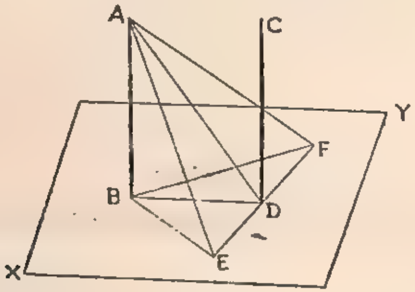
6. Of all straight lines drawn from an external point to a plane, prove that the perpendicular is the shortest.

7. If three points A, B, C on a plane are equidistant from an external point O, show that the foot of the perpendicular from O on the plane is the centre of the circle which can be drawn through A, B, C. [C. U. 1946]

THEOREM 4

If, of two parallel straight lines, one is perpendicular to a plane, the other is also perpendicular to the same plane.

Let the two parallel straight lines AB and CD meet the plane XY in B and D and let AB be perpendicular to the plane.



It is required to prove that CD is also perpendicular to the plane XY.

Join BD and through D draw EDF in the plane XY perpendicular to BD making $DE=DF$.

Join BE, BF and AE, AD, AF.

Since by construction, BD is the perpendicular bisector of EF, we have $BE=BF$.

Now, in the triangles ABE, ABF,

$BE=BF$, AB is common

and $\angle ABE=\angle ABF$, each being a right angle

(Since AB is perpendicular to the plane XY and BE, BF meet it in this plane)

$\therefore$ The triangles are congruent and so $AE=AF$.

Again in the triangles ADE, ADF

since $AE=AF$, AD is common and $DE=DF$.

$\therefore$ $\angle ADE=\angle ADF$ (by construction)

and being adjacent angles, each is a right angle.

Hence DE is perpendicular to DA.

Also, DE is perpendicular to DB (construction).

$\therefore$ DE is perpendicular to the plane in which DA, DB lie, *i.e.*, to the plane of the parallels AB, CD.

Hence CD is perpendicular to DE. ... (1)

Again, since AB and CD are parallel and BD meets them,

$\therefore$ $\angle ABD + \angle CDB = 2$ right angles.

But $\angle ABD = 1$ right angle, since AB is perpendicular to the plane XY.

$\therefore$ $\angle CDB$ is also a right angle

i.e., CD is perpendicular to DB. ... ... (2)

$\therefore$ From (1) and (2), CD is perpendicular to the plane of DE, DB,

i.e., CD is perpendicular to the plane XY.

Exercises

1. Prove that if two straight lines are both perpendicular to the same plane they are parallel to one another. [*C. U.* 1957, '49, '45]

[This is *Converse to theorem 4.*

With the same construction as in the theorem we prove that

DE is perpendicular to DA

Also, DE is perpendicular to DB (construction)

Again, DE is perpendicular to CD (Since, by hypothesis, CD is perpendicular to the plane XY)

$\therefore$ DA, DB and CD are coplanar.

A and B being points in this plane, it follows that AB and CD are coplanar.

And since (by hypothesis)

$$\angle ABD + \angle CDB = 1 \text{ rt. angle} + 1 \text{ rt. angle}$$
$$= 2 \text{ rt. angles.}$$

$\therefore$ AB is parallel to CD.]

2. Prove that straight lines in space which are parallel to a given straight line are parallel to one another.

3. If two intersecting straight lines are respectively parallel to two other intersecting straight lines in a different plane, then prove that the angle contained by the first pair of lines is equal to the angle contained by the second pair.

4. Prove that the figure formed by joining the middle points of the sides of a skew quadrilateral is a parallelogram.

[If the extremities of a pair of skew lines be joined, the figure thus formed is a quadrilateral of which two adjacent sides lie in one plane and the remaining two in another. Such a quadrilateral is known as a **Skew quadrilateral or gauche.**]

5. If perpendiculars are drawn from any point to a system of parallel straight lines in space, then all the perpendiculars lie on a plane perpendicular to the parallel lines. [C. U.]

6. Prove that the locus of the feet of the perpendiculars drawn from an external point to a system of parallel straight lines lying in a plane is a straight line perpendicular to the parallel lines.

7. AB is drawn perpendicular to a plane XY and from B, the foot of the perpendicular, a line BD is drawn perpendicular to any line DE in the plane. Prove that AD is perpendicular to DE.

[**Hint** : Produce ED to F making $DF=DE$ (Fig. Theorem 4). Join BE, BF and AE, AF. Now proceed as in the theorem.

Otherwise : Join BE, AE.

Now $DE^2 = BE^2 - BD^2$ ($\because$ BD is perp. to DE)

$= (AE^2 - AB^2) - (AD^2 - AB^2)$

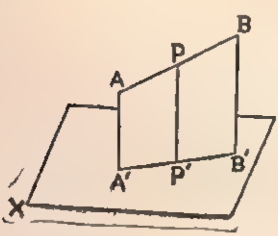
($\because$ AB is perp. to BE and BD)

$= AE^2 - AD^2$

i.e., $AD^2 + DE^2 = AE^2$. $\therefore$ AD is perp. to DE.

This is known as *the Theorem of the Three Perpendiculars*]

8. *Prove that the projection of a straight line on a plane is itself a straight line.* [C. U. 1955]



[Let AB be the given straight line and XY the given plane and let P' be the foot of the perpendicular drawn from *any* point P in AB to the plane XY. Then the locus of P' is the projection of AB.

Let AA' and BB' be drawn perpendiculars from A and B to the plane XY.

Since AA', PP', BB' are all perpendiculars to the plane XY, they are parallel to one another (Converse to Theor. 4).

Since these parallel lines are intersected by the straight line AB, they are coplanar.

$\therefore$ P' lying in the plane of these parallels *viz.* AB' and also in the plane XY, must be a point on the line of section $A'B'$ of these two planes.

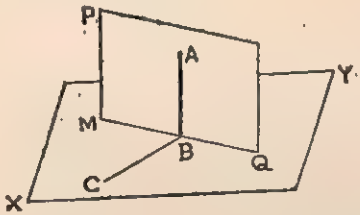
But P being *any* point on the given line, P' is *any* point on the locus.

$\therefore$ The locus of P' *i.e.*, the projection of AB is the straight line $A'B'$.]

THEOREM 5

If a straight line is perpendicular to a plane, then every plane passing through it is also perpendicular to that plane.

Let the straight line AB be perpendicular to the plane XY, and PQ any plane passing through AB.



It is required to prove that the plane PQ is perpendicular to the plane XY.

Let MQ be the line of section of the planes PQ and XY. Draw BC in the plane XY perpendicular to MQ at B.

Now AB, being perpendicular to the plane XY, is perpendicular to MQ, the line of section.

Thus AB, BC lying in the planes PQ, XY respectively are both perpendicular to the line of section MQ.

$\therefore$ $\angle ABC$ measures the dihedral angle between the planes PQ and XY.

Again, since AB is perpendicular to the plane XY and BC meets it in this plane,

$\therefore$ $\angle ABC$ is a right angle.

Hence, the angle which measures the dihedral angle formed by the planes being a right angle, the plane PQ is perpendicular to the plane XY.

Cor. 1. If two planes are perpendicular to one another, then any line drawn in one of the planes perpendicular to the line of section of the planes is also perpendicular to the other plane.

[In the figure of Theorem 5, if PQ and XY be two planes perpendicular to one another, then any line AB drawn in the plane PQ perpendicular to the line of section MQ, is also perpendicular to the plane XY

This can easily be proved by drawing BC in the plane XY perpendicular to MQ at B. For then, the planes being at right angles, $\angle ABC$ is a right angle; also $\angle ABQ$ is a right angle. $\therefore$ AB is perpendicular to the plane of BC, BQ, *i.e.* to the plane XY.]

Cor. 2. If two planes are perpendicular to one another, then a perpendicular drawn from any point on one of the planes to the other plane lies in the first plane.

[Let PQ and XY be two planes perpendicular to one another. If possible, let AD be the perpendicular drawn from any point A in the plane PQ to the plane XY, the foot of the perpendicular D not lying on the line of section MQ. Let AB be drawn perpendicular to MQ. Join BD. Then, by Cor. 1, AB being perpendicular to the plane XY, $\angle ABD$ is a right angle. Also AD being perpendicular to the plane XY, $\angle ADB$ is a right angle. Thus in the triangle ABD, the angles at B and D are right angles which is absurd. Hence D must coincide with B, *i.e.*, the perpendicular drawn from any point A in the plane PQ to the plane XY must lie in the plane PQ.]

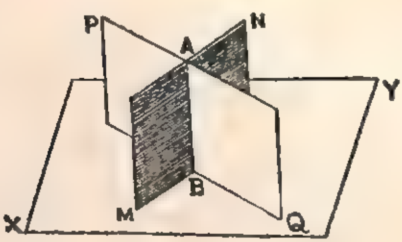
Exercises

1. Draw a plane perpendicular to a given plane and passing through a given straight line not lying in the given plane. [*C. U.* 1954]

2. Through a given point draw a plane perpendicular to each of two intersecting planes. [*C. U.* 1953]

3. If two intersecting planes are each perpendicular to a third plane, prove that their line of section is also perpendicular to that plane. [*C. U.* 1953, '52]

[Let the two planes PQ and MN intersecting along the straight line AB be each perpendicular to the plane XY.



Let a perpendicular be drawn from any point A in the line of section to the plane XY.

Then since the plane PQ is perpendicular to the plane XY and A lies in PQ, the perpendicular drawn also lies in PQ. (Theor. 5, Cor. 2)

Similarly, the perpendicular drawn lies in the plane MN.

Hence the perpendicular drawn lying in both the planes PQ and MN must coincide with AB, the line of section of the planes.

i.e., AB is perpendicular to the plane XY.]

4. A straight line intersects a number of parallel planes ; prove that the angles which the line makes with the planes are all equal.

5. If a plane intersects two parallel planes, show that the corresponding dihedral angles are equal. [*C. U.*]

6. From a point outside two intersecting planes two perpendiculars are drawn on the planes. Show that the line of section of the planes is perpendicular to the plane in which the two perpendiculars lie. [*C. U.*]

CHAPTER III

SOLID FIGURES

1. **Definition :** A **Solid figure** or simply a **solid** is a portion of three dimensional space bounded by one or more surfaces, plane or curved. The bounding surfaces are called **Faces**, and the lines, straight or curved, in which these faces intersect are called **Edges** and the points where any two edges meet are called **Vertices** of the solid.

2. **Polyhedron :** When the faces of a solid are several plane surfaces, it is called a polyhedron.

It can be seen that at least four planes are necessary to enclose a three dimensional space and so the least number of faces of a polyhedron is four. The shape of a polyhedron will vary according to the number and nature of the faces composing it.

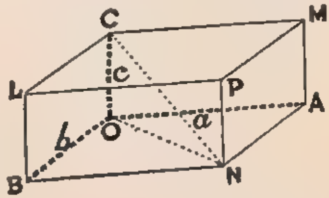
3. **Parallelopiped :** A polyhedron formed by three pairs of parallel planes is called a parallelopiped. The opposite parallel faces of a parallelopiped are congruent parallelograms and it has twelve edges which fall into three groups of four equal straight lines, and eight vertices.

4. **Rectangular parallelopiped** or **Cuboid :** If the faces of a parallelopiped are all rectangles, the figure is a rectangular parallelopiped. It may therefore be defined as a *polyhedron bounded by three pairs of rectangles in parallel planes.*

A rectangular parallelopiped has therefore three pairs of congruent rectangles in parallel planes as faces, three groups of four equal straight lines as edges and eight vertices.

At each vertex three faces mutually at right angles meet and hence the three edges passing through each vertex also form a system of three mutually perpendicular straight lines which are its length, breadth and height. From the adjoining figure it can be easily seen that

(*i*) Any edge is normal to the two faces which it intersects ; *e.g.*, OC is perpendicular to the faces $OBNA$ and $CLPM$.



(*ii*) Any face is perpendicular to the four faces which it meets ; *e.g.*, the face $OCLB$ is perpendicular to the faces $OAMC$, $BNPL$, $CLPM$ and $OBNA$.

(*iii*) If the length OA, the breadth OB and the height OC of a rectangular parallelopiped be a, b, c units respectively, then

(*a*) *the* **whole surface**

$$=2 . \text{ area } OCLB+2 . \text{ area } OAMC+2 . \text{ area } OBNA$$
$$=2bc+2ca+2ab$$
$$=2(bc+ca+ab) \text{ units of area}$$
$=2\times$ *sum of the products of the edges taken two at a time.*

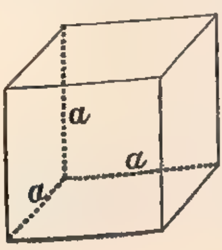
(*b*) *the* **volume** $=abc$ units of volume
$=$ *length* $\times$ *breadth* $\times$ *height*

(*c*) *length of any* **diagonal,** say CN

$$=\sqrt{OC^2+ON^2}$$
$$=\sqrt{OC^2+OA^2+AN^2}$$
$$=\sqrt{OA^2+OB^2+OC^2}$$
$$=\sqrt{a^2+b^2+c^2}.$$

Note : Since OC is normal to the face $OANB$, it is perpendicular to ON which meets it in the plane and so $\angle CON$ is a right angle ; similarly $\angle OAN$ is a right angle, OA being perpendicular to the plane $ANPM$.

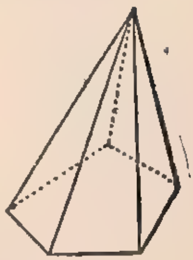
5. **Cube :** If the faces of a rectangular parallelopiped are all squares, the figure is a cube. It may therefore be defined as *a polyhedron bounded by six equal squares.*



A cube has therefore six square faces all of equal area, twelve edges of equal length and eight vertices.

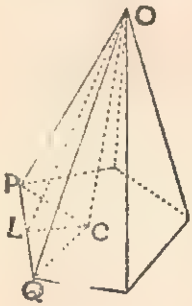
8. Pyramid : A polyhedron having one of its faces called the **base** a polygon of any number of sides and the others all triangles having a common vertex is called a pyramid.

The common vertex of the triangular faces is called the **vertex** of the pyramid and the perpendicular drawn from the vertex to the base is called the **height** of the pyramid.



A pyramid is said to be **rectangular, square** or **triangular** according as the base is a rectangle, a square or a triangle. A triangular pyramid is also called a **tetrahedron.**

9. Right Pyramid : A pyramid having for its base a regular polygon and of which the vertex lies on the straight line drawn perpendicular to the base from its centre (*i.e.*, the centre of the inscribed or circumscribed circle) is called a right pyramid.



It follows that in a right pyramid

(*i*) the side-edges are all equal in length (the two $\triangle^s OCP$ and OCQ are congruent)

(*ii*) the side-faces are all congruent isosceles triangles, and

(*iii*) the altitudes of all the triangular faces are equal. This altitude is called the **slant height** of the pyramid. (OL in the figure.)

If 'a' be the length of each side of the base, 'l' the slant height and 'h' the height of the pyramid, then

(*a*) the **slant surface** $= n \times$ area of a triangular face

$$= n \times \tfrac{1}{2}al$$
$$= l \times \tfrac{1}{2}na$$

(where n is the number of sides of the base)

(*b*) the **volume** $= \frac{1}{3} \times$ area of the base $\times h$.

Hence,

The whole surface = the slant surface + area of the base

The slant surface = semi-perimeter of the base × slant height

The volume = $\frac{1}{3}$ × area of the base × height.

If the length of an edge of a cube be 'a' units, then clearly,

(a) *the whole* **surface** $=6a^2$ units of area

$=6\times(\textit{any edge})^2$

(b) *the* **volume** $=a^3$ units of volume

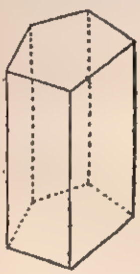
$=(\textit{any edge})^3$

(c) *length of any* **diagonal** $=\sqrt{3}a$

$=\sqrt{3}\times(\textit{any edge})$.

6. **Prism** : A polyhedron having two of its faces called the **end-faces** or **ends** congruent figures in parallel planes and the others called **side-faces** all parallelograms, is called a prism.

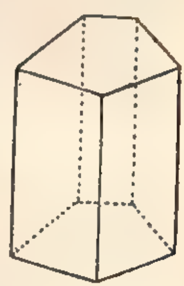
The perpendicular distance between the 'ends' is called the **height** of the prism.



A prism is said to be 'triangular', 'quadrilateral' or 'polygonal' according as the end-faces are triangles, quadrilaterals or polygons having any number of sides.

Since the side-faces of a prism are parallelograms, it follows that any two side-edges of a prism are parallel and equal.

7. **Right Prism** : If the side-edges of a prism are perpendicular to the 'ends', it is called a right prism. It follows that the side-faces of a right prism are rectangles and its height is equal to the length of a side-edge.



If the ends of a right prism are polygons of 'n' sides whose lengths are $a, b, c, \ldots\ldots k$ units and its height be 'h', then clearly

(a) *the whole* **surface**

$=$ lateral surface $+$ area of the two 'ends'

$=(ah+bh+ch+\cdots+kh)+2\times$(area of an end-face)

$=(a+b+c+\cdots+k)h+2\times$(area of an end-face)

$=(\textit{perimeter of the base})\times \textit{height}$

$+2\times(\textit{area of an end-face})$.

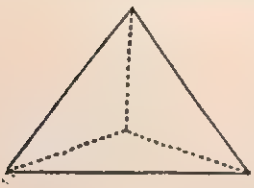
(b) *the* **volume** $=\textit{area of the base}\times\textit{height}$.

Note : It can be seen that a parallelopiped is a prism of which the ends are parallelograms and that a cuboid or a cube is only a special form of a right prism.

10. Tetrahedron : A polyhedron formed by four triangular faces is called a tetrahedron.

A tetrahedron is thus a pyramid on a triangular base.

A tretrahedron is said to be **right** when the base is an equilateral triangle and the three side-faces are congruent isosceles triangles.

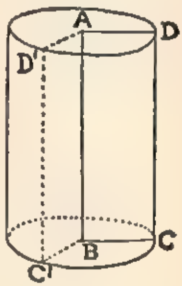


If the base as well as the three side-faces are equilateral triangles, it is a **Regular** tetrahedron.

SOLIDS OF REVOLUTION

11. Right circular cylinder : The solid generated by one complete revolution of a rectangle about one of its sides as axis is called a right circular cylinder.

In the adjoining figure, the rectangle $ABCD$ revolves about the side AB as axis and the solid thus formed is bounded by a curved surface generated by the motion of the side CD and the two plane circular ends described by the sides AD and BC. The side CD which during the motion always remains parallel to the axis is called the **generating line** of the surface and the two plane-ends are called the **bases**. The length of the axis AB is called the **height** of the cylinder. This height is also the distance between the two circular bases.



Note. If a straight line moves parallel to itself and always intersects a fixed curve not in the same plane with the line, the surface generated is said to be **cylindrical.** The fixed curve is called the **Guide** and the moving straight line is called the **generating line** of the curved surface.

In the case of a right circular cylinder the guide is a circle and the plane of the circle is perpendicular to the generating line.

If 'r' be the radius of the base and 'h' the height of the right circular cylinder, then

(a) *the* **curved surface** $=$ *circumference of the base* $\times$ *height*

$= 2\pi rh$ units of area.

(*b*) *the* **whole surface** = *the curved surface + area of the plane ends.*

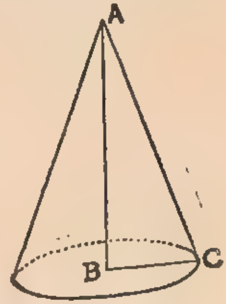
$$= 2\pi rh + 2\pi r^2$$
$$= 2\pi r(h+r) \text{ units of area}$$

(*c*) *the* **volume** = *area of the base × height*

$$= \pi r^2 h \text{ units of volume.}$$

12. Right Circular Cone : The solid generated by one complete revolution of a right-angled triangle about one of the sides containing the right angle as axis is called a right circular cone.

If the right-angled triangle ABC revolve about one of its sides, say AB, containing the right angle, the solid formed is bounded by a curved surface generated by the motion of the hypotenuse AC and the plane circular end described by the side BC. The side AC which during the motion always passes through the point A is called the **generating line** of the surface and the plane circular end is called the **base** of the cone. The point A is called the **vertex** and the angle BAC is called the **semi-vertical angle** of the cone. The length of the axis AB is called the **height** of the cone and the length of the hypotenuse AC is called the **slant height.**



Note. If a straight line moves so as always to pass through a fixed point and intersects a fixed curve, called the guide, not in the same plane with it, the surface generated is said to be **conical**. In the case of a right circular cone the guiding curve is a circle and the vertex lies on the line through the centre of the circle perpendicular to its plane.

If 'r' be the radius of the base, 'h' the height and 'l' the slant height, then

(*a*) *the* **curved surface** = $\frac{1}{2}$ × (*circumference of the base*) × *slant height*

$$= \tfrac{1}{2}.2\pi r.l$$
$$= \pi rl \text{ units of area.}$$

(b) *the* **whole surface** = *the curved surface* + *area of the base*

$$= \pi rl + \pi r^2$$
$$= \pi r(l + r) \text{ units of area.}$$

(c) *the* **volume** = $\frac{1}{3} \times$ (*area of the base*) $\times$ *height*

$$= \tfrac{1}{3}\pi r^2 h \text{ units of volume.}$$

Note. Volume of a cone = $\frac{1}{3} \times$ (volume of the cylinder of the same height and radius of the base).

Def : If a cone be cut off by a plane parallel to its base, then the portion between the base and the parallel plane is called a **Frustum** of the cone. The plane ends are called the **bases** and the perpendicular distance between the bases is called the **height** of the frustum.

13. Sphere : The solid generated by one complete revolution of semi-circle about its diameter as axis is called a sphere.

In the adjoining figure, the semi-circle APB having its centre at O revolves about the diameter AB and generates the sphere. The surface of the sphere is described by the motion of the semi-circumference APB.

It follows that all points on the surface are at a constant distance from the centre O. Hence the sphere may also be defined as *a solid bounded by a curved surface which is the locus of a point which moves in space so that its distance from a fixed point is always the same.* The fixed point is called the **centre** of the sphere and the constant distance is its **radius**.

If 'r' be the radius of a sphere, then

(a) *its* **surface** = $4\pi r^2$ units of area

(b) *its* **volume** = $\frac{4}{3}\pi r^3$ units of volume.

WORKED OUT EXAMPLES

Ex. 1. *The whole surface of a rectangular parallelopiped is* 192 *sq. in. The area of the base and one of the vertical faces are respectively* 48 *sq. in. and* 36 *sq. in. Find the edges.*

If the length, breadth and height be respectively a, b and c, then we have

$$2(ab+bc+ca)=192,\ ab=48 \text{ and } ac=36.$$

Hence $\qquad bc=96-48-36=12$

$\therefore \qquad ab.ac.bc=48.36.12$

$$=(12.4)(12.3)(4.3).$$

i.e., $\qquad (abc)^2=(12.4.3)^2.$

Hence $abc=12.4.3$, considering only positive values.

Dividing respectively by bc, ca and ab, we get

$$a=12,\ b=4 \text{ and } c=3.$$

Ex. 2. *A right prism stands on a triangular base whose sides are* 17 *cm.,* 10 *cm, and* 9 *cm. ; and the height is* 10 *cm. Find the volume and the surface.* [*G. U.*]

We have, area of the triangular base

$$=\sqrt{s(s-a)(s-b)(s-c)}$$

$$\left(\text{where } s=\text{semi-perimeter}=\frac{17+10+9}{2}=18\right)$$

$$=\sqrt{18.1.8.9}=\sqrt{9^2.4^2}=36 \text{ sq. cm.}$$

$\therefore$ Volume $=(\text{area of the base})\times(\text{height})$

$$=36\times10$$

$$=360 \text{ cu. cm.}$$

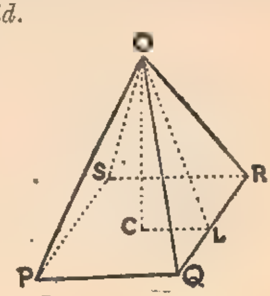
The whole surface = lateral surface + area of the plane ends

$$=(17+10+9)\times10+2\times36$$

$$=360+72$$

$$=432 \text{ sq. cm.}$$

Ex. 3. *A right pyramid of height* 12 *cm. stands on a square base whose side is* 10 *cm. Find (i) the slant edge, (ii) the slant surface and (iii) the volume of the pyramid.*



In the right pyramid $(O, PQRS)$, we have

$$OC=12 \text{ cm.}; \ CL=\tfrac{1}{2}PQ=5 \text{ cm.}$$

$\therefore \quad OL^2=OC^2+CL^2$

$(\because OC$ is perp. to $CL)$

$$=12^2+5^2$$

$$=13^2.$$

Hence (*i*) any slant edge (say OQ)

$= \sqrt{OL^2+LQ^2}$ $\quad$ ($\because$ OL is perp. to QR)

$= \sqrt{169+25}$

$= \sqrt{194}$

$= 13{\cdot}92$ cm.

(*ii*) the slant surface

$= \frac{1}{2}$(perimeter of the base) $\times$ slant height

$= \frac{1}{2}\times 40\times 13$

$= 260$ sq. cm.

(*iii*) the volume $= \frac{1}{3}$ (area of the base) $\times$ height

$= \frac{1}{3}\times 10^2\times 12$

$= 400$ cu. cm.

Ex. 4. *The whole surface of a cylinder is 924 sq. cm. If the diameter of its base be equal to the height, find its volume.* $(\pi=\frac{22}{7})$

If h be the height and r the radius of the base, then the whole surface $= 2\pi rh+2\pi r^2$

$= 2\pi r\cdot 2r+2\pi r^2$ $\quad$ ($\because$ $h=2r$)

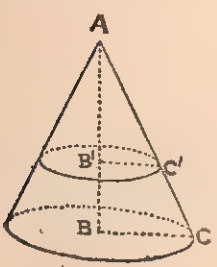
$= 6\pi r^2$

Hence $\quad 6\pi r^2=924$

$\therefore \quad {}^2=\dfrac{924\times 7}{6\times 22}=7^2$

$\therefore$ The volume $=\pi r^2 h=\frac{22}{7}\cdot 7^2.14=2156$ cu. cm.

Ex. 5. *A right circular cone is divided into two parts by a plane parallel to the base. If the volumes of the two portions are equal, prove that the plane divides the axis of the cone in the ratio* $\sqrt[3]{2}-1:1$.



Let $B'C'$ be the radius of the circle in which the dividing plane cuts the cone and let

$AB=h,\ BC=r$

and $\quad AB'=h',\ B'C'=r'$

Then from the given condition,

$\frac{1}{3}\pi r^2 h=2\cdot\frac{1}{3}\pi r'^2 h'$

or, $\left(\dfrac{r}{r'}\right)^2 \dfrac{h}{h'}=2 \quad \cdots \quad (1)$

But from the similar triangles ABC and $AB'C'$

$$\frac{r}{r'}=\frac{h}{h'}.$$

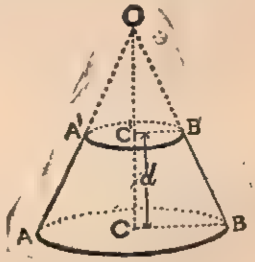
Hence from (1), $\left(\frac{h}{h'}\right)^3=2$

$$\therefore \quad \frac{h}{h'}=\sqrt[3]{2}$$

$$\therefore \quad \frac{h-h'}{h'}=\frac{\sqrt[3]{2}-1}{1}$$

i.e., the plane divides the axis of the cone in the ratio $\sqrt[3]{2}-1:1$.

Ex. 6. *Find the volume of the frustum of a cone the radii of whose bases are r_1 and r_2, the height of the frustum being d.*



Let $OC=h_1$, $CB=r_1$

and $OC'=h_2$, $C'B'=r_2$.

Then from similar triangles,

$$\frac{r_1}{h_1}=\frac{r_2}{h_2}=k \text{ (say)} \quad \cdots \quad \text{(i)}$$

The required volume

$=$ vol. of the cone OAB $-$ vol. of the cone $OA'B'$

$=\frac{1}{3}\pi r_1{}^2h_1-\frac{1}{3}\pi r_2{}^2h_2$

$=\frac{1}{3}\pi(r_1{}^2h_1-r_2{}^2h_2)$

$=\frac{1}{3}\pi k^2(h_1{}^3-h_2{}^3)$ from (i)

$=\frac{1}{3}\pi k^2(h_1-h_2)(h_1{}^2+h_1h_2+h_2{}^2)$

$=\frac{1}{3}\pi d(k^2h_1{}^2+kh_1.kh_2+k^2h_2{}^2)$ [$\because$ $h_1-h_2=d$]

$=\frac{1}{3}\pi d(r_1{}^2+r_1r_2+r_2{}^2)$.

Ex. 7. *A lump of clay in the form of a solid sphere is converted into a right circular cylinder of height 16 inches. Find the radius of the base of the cylinder supposing it to be equal to the radius of the sphere.* [*C. U.*]

If r be the radius of the sphere then the radius of the base of the cylinder is also r. We have then

$\frac{4}{3}\pi r^3=\pi r^2.16$ since the volumes are equal.

$\therefore$ $r=12$ inches.

Exercises

1. The diagonal of a rectangular parallelopiped is 13 inches and the whole surface is 192 sq. inches. Find the sum of the three sides.

2. A vertical pillar 12 ft. high stands on a rectangular base whose area is 12 sq. ft. If its diagonal is 13 ft., find the dimensions of the base.

3. The length, breadth and height of a rectangular block are proportional to 3, 4 and 5. If the whole surface of the block is 2350 sq. cm., find the dimensions of the block.

4. What is the length of the edge of a cube of which the total area of the surface is 346·56 sq. cm.? [*C. U.* 1956]

5. The length, breadth and height of a closed box are 12 in., 10 in. and 8 in. respectively and the total inner surface is 376 sq. in.; if the walls of the box are uniformly thick, find the thickness. [*C. U.* 1958]

6. Find the volume and the lateral surface of a right prism 8 inches long standing on an isosceles triangle, each of whose equal sides is 5 in. and the other side 6 in. [*C. U.* 1958]

7. The base of a right prism is a right-angled triangle, the sides containing the right angle being 3 cm. and 4 cm. If its height be 5 cm., find the volume and the lateral surface.

8. The base of a right prism of height 12 cm. is a triangle whose perimeter is 36 cm., and the volume is 432 cu. cm.; find the radius of the inscribed circle of the base triangle.

9. Find the volume of the pyramid of which

(*i*) the base is a triangle whose sides are 8 cm., 15 cm. and 17 cm. and the height is 12 cm. [*C. U.*]

(*ii*) the base is a triangle of sides 5 in., 6 in. and 7 in. and the height is $5\sqrt{6}$ in.

10. Find to the nearest hundredth of a centimetre the slant edge of a right pyramid whose height is 12 cm. and which stands on a square base of side 8 cm.

11. The base of a right tetrahedron is an equilateral triangle of side nches and its height is 4 inches. Find the slant height and the surface area.

12. The faces of a tetrahedron are four equilateral triangles; find the area of the faces of the tetrahedron, if the length of a side of each triangle is 4 ft. Find also the volume of the tetrahedron. [*C. U.*]

13. The diameter of the base of a cylinder is 18 inches and its curved surface is 450 sq. inches. Find the volume of the cylinder.

14. A cubic inch of gold is drawn into a wire 1000 yards long; find the diameter of the wire to the nearest thousandth of an inch. ($\pi=3\cdot1416$) [C. U. 1958]

15. If h be the height and α the semi-vertical angle of a right circular cone, then prove that its surface S and the volume V are given by

$$S=\frac{\pi h^2 \tan \alpha}{\cos \alpha},\ V=\tfrac{1}{3}\pi h^3 \tan^2\alpha$$

16. Find the volume and the area of the slanting surface of a right circular cone of height 4 feet and the radius of whose base is 3 feet ($\pi=\frac{22}{7}$) [C.U.]

17. Show how to draw a plane parallel to the base of a right circular cone so that it divides the cone into two parts of equal surfaces. [C. U.]

18. A right circular cone 20 feet high has its upper part cut off by a plane passing through the middle point of its axis. If the plane of section be at right angles to the axis, and if the radius of the base of the original cone be 4 feet, find the volume of the truncated cone. ($\pi=\frac{22}{7}$) [C. U.]

19. How many solid circular cylinders each of length 8 inches and diameter 6 inches can be made out of a solid sphere of radius 6 inches? [C. U. 1952]

20. The volume of a sphere is twice the area of its surface. Find the radius of the sphere. [C. U. 1953]

21. Three solid spheres of glass whose radii are 1 cm., 6 cm. and 8 cm. respectively are melted into a single solid sphere. Find the radius of the sphere so formed. [C. U. 1958]

22. How many solid spheres, each 6 cm. in diameter could be moulded from a solid metal cylinder whose length is 45 cm. and diameter 4 cm.? If the cylinder of the above dimensions be hollow, how many circular discs of diameter 6 cm. may be made out of it? [C. U. 1950]

23. A sphere and a right circular cylinder of the same radius have equal volumes. By what percentage does the diameter of the cylinder exceed its height? [C. U.]

24. A cylinder, a hemisphere and a cone have equal bases and are of the same height; compare their volumes.

Answers :

1. 19. 2. 4 ft. ; 3 ft. 3. 15 cm., 20 cm., 25 cm. 4. 7·6 cm.
5. 1 cm. 6. 96 cu. in., 128 sq. in. 7. 30 cu. cm., 60 sq. cm.
8. 2 cm. 9. (*i*) 240 cu. cm. ; (*ii*) 60 cu. in. 10. 13·26 cm.
11. 4·04 in. ; 13·85 sq. in. 12. $4\sqrt{3}$ sq. ft. ; $\frac{10}{3}\sqrt{2}$ cu. ft.
13. 2025 cu. in. 14. 0·006 in. 16. $37\frac{5}{7}$ cu. ft., $47\frac{1}{7}$ sq. ft.
17. $\sqrt{2}-1 : 1$. 18. $293\frac{1}{3}$ cu. ft. 19. 4. 20. 6. 21. 9.
22. 5 ; 20. 23. 50%. 24. 3 : 2 : 1.

UNIVERSITY AND BOARD EXAMINATION QUESTIONS

C. U. Pre-University Examination

1961

1. (a) The co-ordinates of A, B, C are $(-1, 5)$, $(3, 1)$ and $(5, 7)$ respectively. D, E, F are the middle points of BC, CA, AB respectively. Calculate the area of the triangle DEF.

(b) Obtain the equation of the straight line through the point $(2, 1)$ and perpendicular to the line joining the points $(2, 3)$ and $(3, -1)$.

2. (a) Obtain the equation of the locus of a point which moves in the plane of (xy) in such a way that its distance from the point $(2, 3)$ is always two-thirds of its distance from the y-axis.

(b) Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x', y').

3. (a) Show that the centres of the following three circles are in a straight line:

$$x^2+y^2-2x-6y-5=0, \; x^2+y^2-4x-10y-7=0$$
$$x^2+y^2-6x-14y-9=0$$

(b) Find the eccentricity and the coordinates of the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$

4. (a) Obtain the equations of the lines which bisect the angles between the lines

(i) $a_1x+b_1y+c_1=0$ and (ii) $a_2x+b_2y+c_2=0$.

(b) Obtain the equation of the circle which has its centre at the point $(3, 4)$ and touches the straight line $5x+12y=1$

5. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, show that it is also perpendicular to the plane in which they lie.

6. (a) Find the locus of a point in space equidistant from two given points.

(b) Three solid spheres of gold whose radii are 1 cm., 6 cms., and 8 cms. respectively are melted into a single gold sphere. Find the radius of the sphere so formed.

1962

1. (a) Find the angle between the two straight lines

$$y=mx+c \text{ ; and } y=m'x+c'.$$

(b) Obtain the equations to the straight lines each of which passes

through the point $(2, -1)$ and intersects the axes of coordinates at points equidistant from the origin and calculate the angle between them.

2. (a) Obtain the equation to the circle which passes through the points $(2, -1)$ and $(3, -2)$ and has its centre on the straight line

$$2x+4y-3=0.$$

(b) A conic is represented by the equation $4x^2-9y^2=36$; Calculate its eccentricity, length of latus rectum and the coordinates of the foci.

3. (a) Show that the chord of the parabola $y^2=4ax$, whose equation is $y-x\sqrt{2}+4a\sqrt{2}=0$, is a normal to the parabola, and find the coordinates of the point of the parabola at which it is normal.

(b) Find possible values of k for which the straight line $3x+4y=k$ may touch the circle $x^2+y^2=10x$.

4. (a) Find the equations to the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the point (x', y').

(b) A point P moves in the plane of (xy) in such a way that its distance from the lines $12x+5y-4=0$ and $3x+4y+7=0$ are equal. Obtain the equation to the locus traced out by P.

5. (a) Show that all straight lines drawn perpendicular to a given straight line at a given point on it are coplanar.

(b) The diagonal of a rectangular block is 10 cms., and the sum of the lengths of its edges is 80 cms. Calculate the total area of the outer surface of the block.

6. (a) Show that if a straight line is perpendicular to a plane, then every plane passing through it is also perpendicular to that plane.

(b) A right circular cone is 10 cms. high and its slant height is 15 cms. Calculate the volume of the cone.

[$\pi=22/7$]

1963

1. (a) Find the area of the triangle the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3).

(b) If (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the coordinates of the vertices A, B, C respectively of the triangle ABC and if D is the middle point of BC, find the coordinates of G which divides AD such that $AG=2GD$.

2. (a) Find the equation of the straight line passing through the point $(3, 5)$ and parallel to the line $4x-3y+1=0$.

(b) Find the equations of the tangents to the circle $x^2+y^2=9$ which are parallel to the line $3x+4y=0$

3. (a) Find the locus of the point whose distance from the point $(-1, 1)$ is equal to its perpendicular distance from the straight line $x+y+1=0$

(b) Find the co-ordinates of the focus of the parabola $y^2=4ax$ which passes through the point of intersection of the lines

$$\frac{x}{3}+\frac{y}{2}=1 \text{ and } \frac{x}{2}+\frac{y}{3}=1$$

4. (a) If the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be equal to distance between the foci, find the eccentricity of the ellipse.

(b) Find the equation of the ellipse referred to its centre as origin and major axis as x-axis, whose latus rectum is 5 and eccentricity is $\frac{2}{3}$.

5. (a) Prove that two intersecting planes cut one another in a straight line and in no point outside it.

(b) If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, show that it is also perpendicular to the plane in which they lie.

6. (a) Three solid gold spherical beads of radii 3, 4, 5 cms. respectively are melted into one solid spherical bead. Find its radius.

(b) The volume of a right circular cylinder and a right circular cone standing on the same base are as 3 : 2. Show that the height of the cone is double the height of the cylinder.

1964

1. (a) The three points, whose co-ordinates are (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie in a straight line. Show that

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$$

(b) Find the angle between the straight lines

$$x-\sqrt{3}y=1 \text{ and } \sqrt{3}x-y=4.$$

2. (a) Find the equation of the circle, having its centre at (1, −2) and passing through the point of intersection of the straight lines

$$3x+y=14 \quad \text{and} \quad 2x+5y=18$$

(b) The angle between the two tangents drawn from a point P to the circle $x^2+y^2=a^2$ is 120°. Prove that the locus of P is $x^2+y^2=4a^2/3$.

3. (a) Obtain the equation to the parabola whose focus is *at* the point (5,0) and whose directrix is the straight line $3x-4y+2=0$.

(b) Find the equation to the normal to the parabola $y^2=4ax$ *at* the point $(am^2, 2am)$.

4. (a) Find the latus rectum, the eccentricity and the coordinates of the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$

(b) Obtain the equations of the tangents of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$, which are parallel to the straight line $\sqrt{3}x-2y=0$

5. (a) Prove that all straight lines, drawn perpendicular to a given straight line *at* a given point on it, are coplanar.

(b) If a triangle revolves about its base, prove that the locus of its vertex is a circle.

6. (a) A right prism stands on a triangular base whose sides are 18″, 20″, 34″. If the height of the prism is 10″, find the area of its total surface.

(b) How many square feet of canvas are required for a conical tent 24 ft. high, the diameter of the base being 14 ft.? [$\pi=\frac{22}{7}$].

1965

1. (a) Find the angle between the straight lines whose equations are $y=m_1x+c_1$ and $y=m_2x+c_2$.

(b) Find the equation of the straight line passing through the point $(-3, 1)$ and perpendicular to the straight line $x-y+3=0$.

2. (a) Find the equation of the circle passing through the points $(0, 0)$, $(4, 0)$ and $(0, 3)$.

(b) Prove that the centres of the three circles $x^2+y^2-4x-6y-5=0$, $x^2+y^2-8x-2y-8=0$ and $x^2+y^2+4x-14y+11=0$ lie in a straight line.

3. (a) Show that the straight line $y=mx+\dfrac{a}{m}$ is a tangent to the parabola $y^2=4ax$.

(b) Show that the locus of the point of intersection of the tangents $y=mx+\dfrac{a}{m}$ and $y=m'x+\dfrac{a}{m'}$ to the parabola $y^2=4ax$ is a straight line, if mm' is constant.

4. (a) Find the equation of the chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, which is bisected at the point (h, k).

(b) Show that the locus of the middle points of a system of parallel chords of an ellipse is a straight line.

5. (a) Find the latus rectum, the eccentricity and the coordinates of the foci of the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$.

(b) The eccentricities of the hyperbolas $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ and $\dfrac{x^2}{b^2}-\dfrac{y^2}{a^2}=1$ are e and e' respectively. Prove that $\dfrac{1}{e^2}+\dfrac{1}{e'^2}=1$.

6. (a) If, of two parallel straight lines, one is perpendicular to a plane, prove that the other is also perpendicular to the plane.

(b) PN is drawn perpendicular to a plane XY from an outside point P, and from the foot N of the perpendicular, a straight line NM is drawn perpendicular to a straight line AB on the plane XY. Prove that PM is perpendicular to AB.

7. (a) How many solid circular cylinders, each of length 8 inches and diameter 6 inches can be made out of the material of a solid sphere of radius 6 inches ?

(b) Find the volume of a pyramid whose base is a triangle having sides 9 cms., 10 cms. and 17 cms., and whose height is 12 cms.

1966

1. (a) Obtain the distance between two points whose co-ordinates are (x_1, y_1) and (x_2, y_2).

(b) Show that the points (?, 3), $(-3, -5)$ and $(-5, -3)$ are the vertices of an isosceles triangle.

2. (a) Prove that the straight line

$$y=mx+a\sqrt{1+m^2}$$

is always a tangent to the circle

$$x^2+y^2=a^2.$$

(b) Find the locus of the point of intersection of two mutually perpendicular tangents to the circle $x^2+y^2=a^2$.

3. (a) Find the equation of the normal to the parabola $y^2=4ax$ in the form

$$y=mx-2am-am^3.$$

(b) The normal to the parabola $y^2=4ax$ at the point $(am^2, -2am)$ meets the parabola again at $(am'^2, -2am')$. Show that

$$m^2+mm'+2=0.$$

4. (a) If S, S' be the two foci of an ellipse and SY, $S'Y'$ perpendiculars from S, S' upon the tangent at any point of the ellipse, prove that

$$SY.S'Y'=b^2,$$

where b = semi-minor axis of the ellipse.

(b) Find the equation of the hyperbola whose focus is (3, 1), directrix the straight line $2x+y-1=0$ and eccentricity $\sqrt{2}$.

5. (a) Prove that two intersecting planes cut one another in a straight line and in no points outside it.

(b) Find the height of a conical tent such that it can accommodate 4 persons each of whom requires 40 sq. ft. of space on the ground and 200 cu. ft. of air to breathe.

6. (a) Prove that all straight lines, drawn perpendicular to a given straight line at a given point on it, are coplanar.

(b) A sphere of radius 10 cms. is dropped into a cylindrical vessel partly filled with water. The radius of the vessel is 20 cms. If the sphere be completely submerged, by how much will the surface of water be raised ?

Burdwan University Entrance Examination

1961

1. (a) Find the equation to the straight line passing through the intersection of the lines $x+2y-3=0$ and $3x+4y+7=0$ and perpendicular to the straight line $y-x=8$.

(b) Show that the three points $(4, 2)$, $(7, 5)$ and $(9, 7)$ are in one straight line and find the equation of the line of collinearity.

2. (a) Find the equation of the circle described on the diameter whose end points are (x_1, y_1) and (x_2, y_2)

(b) Find the equation of the tangent to the circle $x^2+y^2=25$ making an angle of 30° with the straight line $3x+4y=0$.

3. (a) Find the condition that $y=mx+c$ will be a normal to the parabola $y^2=4ax$.

(b) Prove that the tangent to the parabola $y^2=8x$ at a given point on it makes equal angles with the focal distance of the point and its perpendicular distance from the directrix.

4. (a) Find the eccentricity, length of the latus rectum and the coordinates of the foci of the ellipse $16x^2+9y^2=144$.

(b) Prove that the tangent to an ellipse at a point on it makes equal angles with the focal distances of the point.

5. (a) All straight lines drawn perpendicular to a given straight line at a given point on it are coplanar.

(b) Find the volume and the lateral surface of a right prism of height 8 inches standing on an isosceles triangle each of whose equal sides is 5 inches and the other side is 6 inches.

6. (a) If a straight line is perpendicular to a given plane then any plane passing through the line is perpendicular to the given plane.

(b) If a straight line is parallel to a given plane, prove that the line of intersection of any plane through the given line with the given plane is parallel to the given line.

1962

1. (a) Prove analytically that the line joining the middle points of any two sides of a triangle is half the third side.

(b) Find the equations of the medians of the triangle whose vertices are the points $(3, 2)$, $(1, -1)$, $(-19, -9)$ and show that they are concurrent.

2. (a) Find the equations of the tangents to the circle $(x-1)^2+(y+1)^2=16$ parallel to the line $y=3x+10$.

(b) Prove that the points (1, 1), (2, 0), (3, −3), (−5, −7) are concyclic and find the equation of the circle passing through them.

3. (a) Find the equation to the parabola whose focus is at the origin and whose directrix is the straight line $2x+y-1=0$. Find out its vertex.

(b) Show that the tangents to the parabola $y^2=8x$ at the points (2, 4) and (2, −4) intersect on the x-axis. Find also the angle between the tangents.

4. (a) Starting from the focus directrix properties of an ellipse, show that the equation of an ellipse is of the form : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(b) Find the length of the latus rectum, the foci and the equations of the directrices of the hyperbola $3x^2-4y^2=48$.

5. (a) If two straight lines are parallel and if one of them is perpendicular to a plane, show that the other is also perpendicular to that plane.

(b) Prove that all straight lines drawn perpendicular from a given point to a system of parallel straight lines in space are coplanar.

6. (a) How many solid circular cylinders of length 8 in. and diameter 6 in. can be made out of the material of a solid sphere of radius 6 in. ?

(b) A right circular cone 15 cm. high, the radius of the base being 8 cm. has its upper part cut off by a plane through the middle point of its axis parallel to the base. Find the volume of the truncated cone.

1963

1. (a) Show that the area of the triangle, the coordinates of whose angular points are $\left(a,\frac{1}{a}\right)$, $\left(b,\frac{1}{b}\right)$ and $\left(c,\frac{1}{c}\right)$ is, $\frac{(b-c)(c-a)(a-b)}{2abc}$

(b) If the centroid of a triangle is (1, 4) and two of its vertices are (4, −3) and (−9, 7), find the other vertex.

2. (a) Show that the equation

$$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0,$$

where λ is an arbitrary constant, represents any straight line passing through the point of intersection of the lines

$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0.$$

(b) Find the equation of the straight line which passes through (x_1, y_1) and is perpendicular to the join of (x_2, y_2) and (x_3, y_3).

3. (a) Find the equation of the circle which is concentric with the circle $x^2+y^2-4x+6y-3=0$ and which passes through the point $(5,-2)$

(b) If $y=x\sin\alpha+a\sec\alpha$ be a tangent to the circle $x^2+y^2=a^2$, then prove that $\cos^2\alpha=1$

4. (a) Find the condition that the straight line $y=mx+c$ should touch the parabola $y^2=4ax$

(b) Find the equation of the ellipse (referred to its axes as the axes of x and y respectively) which passes through the point $(-3, 1)$ and has the eccentricity $\sqrt{\frac{2}{5}}$.

5. (a) If two intersecting planes are each perpendicular to a third plane, prove that their line of intersection is also perpendicular to that plane.

(b) If, of three lines of intersection of three planes, two be parallel, show that the third will also be parallel to the other two.

6. (a) The length, breadth and the height of a closed box are 12 in., 10 in., and 8 in. respectively and the total inner surface is 367 sq. in. If the walls of the box are uniformly thick, find the thickness.

(b) If S be the area of a closed surface, V be the volume, h the height and α the semi-vertical angle of a right circular cone, prove that

$$S=\frac{\pi h^2\sin\alpha}{\cos^2\alpha}\text{ and }V=\tfrac{1}{3}\pi h^3\tan^2\alpha.$$

1964

1. (a) A point P divides the straight line AB internally in the ratio $\alpha:\beta$. The co-ordinates of A and B are (a_1, a_2) and (b_1, b_2) respectively. Find the coordinates of P.

(b) Find the equation of the locus of a point $P(x, y)$ if the ratio of its distance from the point $(-1, 0)$ to its distance from the point $(1, 0)$ be $2:1$.

2. (a) Find an expression for the angle between the two straight lines given by $Ax+By=0$ and $A'x+B'y=0$ respectively.

(b) Find the distance of the point of intersection of the two straight lines $2x-3y+17=0$ and $3x+4y=0$ from the straight line $4x-3y=0$.

3. (a) Find the equation of the circle described on the diameter whose end-points have the co-ordinates (x_1, y_1) and (x_2, y_2) respectively.

(b) Find the equation of the circle which passes through the points $(1, -2)$ and $(4, -3)$ and which has its centre on the straight line

$$(3x+4y=5)$$

4. (a) Find the equation of the tangent to the parabola $y^2=8x$ which is perpendicular to the straight line $3x-y+7=0$.

(b) Find the distance between the focii of the ellipse whose equation in standard form is $3x^2+4y^2=24$.

5. (*a*) What do you mean by (i) the angle between a plane and a straight line ; (ii) the angle between two planes ?

If a right angle revolves about one of its arms then prove that the other arm describes a plane.

(*b*) The base of a right prism is a triangle whose perimeter is 15 cm. and the radius of the in-circle of the triangle is 3 cm. If the volume of the prism be 270 c.c., find its height.

6. (*a*) Prove that, if a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, then it is perpendicular to the plane in which they lie.

(*b*) Prove that all straight lines drawn perpendicular from a given point to a system of parallel straight lines in space are coplanar.

1965

1. (*a*) Find the area of a triangle whose vertices are the points $A(x_1, y_1)$ $B(x_2, y_2)$ and $C(x_3, y_3)$ respectively.

(*b*) Prove that the four points (2, 1), (4, 3), (2, 5) and (0, 3) joined in order from a parallelogram.

2. (*a*) Prove that the length of the perpendicular drawn from the point (x_1, y_1) upon the straight line $Ax+By+C=0$ is the numerical value of

$$\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$$

(*b*) Find the distance of the origin from the line $-3x+4y+11=0$ measured parallel to the line $2x+y+2=0$.

3. (*a*) Prove that the line $y=mx+\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2-1=0$.

(*b*) Find the equation of a circle whose centre is at the origin and which meets the line $\frac{x}{5}-\frac{y}{6}=1$ on the axis of y.

4. (*a*) Find the vertex and the axis of the parabola whose focus is (3, 4) and directrix is $3x+4y+25=0$.

(*b*) Find the equation of a tangent to the ellipse which makes an angle of 60° with the x-axis, the equation of the ellipse being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

5. (*a*) Prove that all straight lines drawn perpendicular to a given straight iine at a given point on it are coplanar.

(*b*) Prove that a point can be found in a plane equidistant from three points outside the plane, if the three points are non collinear, and lie in a plane which is not perpendicular to the given plane.

6. (*a*) A plane is drawn parallel to the base of a right circular cone so as to divide it into two parts of equal volumes. Show that the plane divides the axis in the ratio $1:(\sqrt[3]{2}-1)$

(*b*) Show that the volume of a right prism of length l and with the end-face an equilateral triangle of side a, is $\sqrt{\frac{3}{4}}a^2l$.

Board Of Secondary Education, West Bengal

Higher Secondary Examination

1960

1. (*a*) Obtain the coordinates of the point which divides the straight line joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m_1 : m_2$

(*b*) If A, B, C, D are points whose coordinates are $(-2, 3)$, $(8, 9)$, $(0, 4)$ and $(3, 0)$ respectively, and AB and CD are joined, find the ratio of the segments into which AB is divided by CD.

(*c*) Obtain the equation of the straight line whose intercepts on the axes OX, OY are a and b respectively.

(*d*) Determine the equation of the straight line which passes through the intersection of the lines given by $3x-4y+1=0$ and $5x+y=1$, and has equal intercepts of the same sign on the axes.

2. (*a*) Find the length of the chord of the circle $x^2+y^2=64$, intercepted on the straight line $3x+4y-c=9$.

(*b*) Obtain the coordinates of the point of contact of any one of the two tangents to the above circle $x^2+y^2=64$, parallel to the line $3x+4y-c=0$.

(*c*) Find out the eccentricity, and the coordinates of the foci of the ellipse $9x^2+25y^2=225$.

(*d*) Find the distance from the origin of the point where the tangent at the extremity of a latus rectum of the above ellipse $9x^2+25y^2=225$, intersects the major axis.

3. (*a*) Find out the equation of the tangent of the parabola $y^2=4ax$ at the extremity of the latus rectum.

(*b*) A double ordinate of the parabola $y^2=4ax$ is of length $8a$. Prove that the lines joining the vertex to its two ends are at right angles.

(*c*) Obtain the equation to the hyperbola whose focus is (a, o), directrix is the straight line $x=\frac{1}{2}a$, and eccentricity is $\sqrt{2}$.

(*d*) A rod of length 6 units slides with its extremities always on the coordinate axes. Prove that its middle point traces out a circle, whose equation you are to determine.

4. (*a*) A thick hollow cylindrical pipe is 6 inches in length, and its whole surface (outer and inner curved surfaces and the plane edges) is 308 sq. inches. If the external diameter of the pipe is 8 inches, and if its material weighs 5 ozs. per cubic inch, find its weight.

[Take $\pi=22/7$]

(b) When is (i) a straight line, (ii) a plane, said to be perpendicular to a given plane ?

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, prove that it is perpendicular to the plane containing them.

(c) Obtain the coordinates of the centre of the circle passing through the points (1, 2), (3, −4), (5, −6) and determine the length of its diameter.

Is the origin inside or outside the circle ?

1961

1. (a) Obtain the area of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

(b) Find the area of the triangle whose vertices A, B, C are respectively (3, 4), (−4, 3) and (8, −6) ; hence, or otherwise find the length of the perpendicular from A on BC.

(c) Obtain the equation of the straight line passing through the points (x_1, y_1) and (x_2, y_2).

(d) Obtain the equation to the perpendicular-bisector of the line joining the points (−2, 7) and (8, −1). At what distance is this perpendicular-bisector from the origin ?

2. (a) Obtain the equation to the circle passing through the points (3, 4), (3,—6), (—1, 2) and determine its centre and radius.

(b) Prove that the straight line $y=x+a\sqrt{2}$ touches the circle $x^2+y^2=a^2$, and find its point of contact.

(c) Obtiain the equation to the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) on the ellipse.

(d) Find the equation to the tangent of the ellipse $9x^2+16y^2=144$ having equal positive intercepts on the axes.

3. (a) Find out the equation to the parabola whose focus is (−3, 4) and directrix is $6x-7y+5=0$

(b) The two tangents drawn from a point P to the parabola $y^2=4x$ are at right angles. Find the locus of P.

(c) In the hyperbola $4x^2-9y^2=36$, find the lengths of the axes, the coordinates of the foci, the eccentricity and the length of the latus rectum.

(d) Find the condition that $y=mx+c$ may touch the hyperbola $x^2-y^2=a^2$.

4. (a) A and B are two fixed points whose coordinates are (2, 4) and (2, 6) respectively ; ABP is an equilateral triangle on the side of AB opposite to the orgin. Find the coordinates of P.

(b) B and C are fixed points having coordinates $(3, 0)$ and $(-3, 0)$ respectively. If the vertical angle BAC be $90°$, show that the locus of the centroid of the triangle ABC is a circle whose equation you are to determine.

(c) With the material of a hollow sphere of outer diameter 10 cms. and thickness 2 cms. is made a solid right circular cone of height 8 cms. Find the surface area of the curved surface to the nearest square centimetre. $[\pi = 22/7]$

(d) How is the angle between two intersecting planes defined? When is a plane perpendicular to another plane?

If two straight lines are parallel, and if one of them is perpendicular to a plane, prove that the other is also perpendicular to the same plane.

1962

1. (a) Find the coordinates of the point which divides in a given ratio $m_1 : m_2$ internally, the line joining two given points (x_1, y_1) and (x_2, y_2).

The coordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3). Find the coordinates of the point where the medians of the triangle intersect.

2. (a) Find the angle between the straight lines whose equations are $y = m_1x + c_1$ and $y = m_2x + c_2$.

(b) Find the equation of the straight line passing through the point $(-3, 1)$ and perpendicular to the line $5x - 2y + 7 = 0$.

3. (a) Find the equation of the circle passing through the origin which makes intercepts 6 and 8 on the positive sides of the axes of x and y respectively.

(b) Prove that the centres of the three circles

$x^2 + y^2 - 2x + 6y = -1$, $x^2 + y^2 + 4x - 12y = 9$ and $x^2 + y^2 - 16 = 0$

lie on a straight line.

4. (a) Find the equation of the parabola, whose focus is at the point $(5, 0)$ and whose directrix is the line $3x - 4y + 2 = 0$

(b) Show that the straight line $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$.

5. (a) Find the equation of the ellipse whose major and minor axes lie along the axes of coordinates x and y respectively and whose eccentricity is $\frac{1}{\sqrt{2}}$ and latus rectum 3.

(*b*) Show that the line $x-y=5$ touches the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

6. Prove that all straight lines drawn perpendicular to a given straight line at a given point are coplanar.

7. If a right angle rotates about one of its arms, then the other arm describes a plane. Prove this.

8. Find the volume and the lateral surface of a right prism 8 inches long, standing on an isosceles triangle, each of whose equal sides is 5 inches and the other side is 6 inches.

9. A right pyramid stands on a rectangular base whose sides are 12 inches and 9 inches ; and the length of each of the slant edges is 8·5 inches, Find the height and the volume of the pyramid.

1963

1. (*a*) Obtain the distance between two points whose rectangular Cartesian coordinates are (x_1, y_1) and (x_2, y_2).

(*b*) Prove that the points $(2, -2)$, $(8, 4)$, $(5, 7)$ and $(-1, 1)$ are the successive angular points of a rectangle.

2. (*a*) Obtain the perpendicular distance from the point (x_1, y_1) to the straight line $ax+by+c=0$

(*b*) Find the ortho-centre of the triangle whose angular points are $(2, 7)$, $(-6, 1)$ and $(4, -5)$.

3. (*a*) Find the equation to the tangent at (x_1, y_1) of the circle

$$x^2+y^2=a^2.$$

(*b*) Obtain the equation to the circle which passes through the point $(0, 4)$ and touches the x-axis at the point $(2, 0)$.

4. (*a*) A tangent to the parabola $y^2=12x$ makes an angle 45° to the axis. Find the coordinates of its point of contact.

(*b*) The coordinates of the foci of a hyperbola are $(5, 0)$ and $(-5, 0)$, and its eccentricity is $\frac{5}{3}$. Find its equation.

5. (*a*) Show that the locus of the middle points of a system of parallel chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a straight line passing through its centre.

(*b*) Find the equation to the normal to the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

at an extremity of a latus rectum.

6. (*a*) If a straight line is perpendicular to each of two intersecting

straight lines at their point of intersection, prove that it is perpendicular to the plane in which they lie.

(b) If $PA = PB = PC$, where P is a point outside the plane of the triangle ABC, and if PO be drawn perpendicular to the plane, prove that O is the circum-centre of the triangle ABC.

(c) If two straight lines are both perpendicular to a plane, show that they are parallel.

(d) If the middle points of the adjacent sides of a skew quadrilateral are joined, prove that the figure so formed is a parallelogram.

7. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of the base is to the height as 3 : 4.

or, A sphere of diameter 6 cms. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cms. If the sphere be completely submerged, by how much will the surface of the water be raised ?

1964

1. Find the coordinates of the point which divides the straight line joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$.

Write down the coordinates of the middle point of the straight line joining the points (7, −4) and (−5, 6).

2. Find the equation of the straight line passing through the intersection of the straight lines $2x - 7y + 11 = 0$ and $x + 3y - 8 = 0$, if it

(a) passes through the origin ;

(b) is perpendicular to the straight line $2x - 5y + 6 = 0$;

(c) makes equal intercepts on the two axes.

3. Prove that the straight line $3x + 4y + 7 = 0$ touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ and find the coordinates of the point of contact.

4. Find the focus, vertex and directrix of the parabola

$$(y+3)^2 = 2(x+2)$$

5. What do you understand by the term 'eccentricity' as applied to a hyperbola ?

Find the equation of the hyperbola whose focus is (2, 3), and directrix, the line $x + 2y = 1$ and eccentricity $\sqrt{3}$.

6. Give instances from the sides and edges of a cube of :

(a) parallel planes, (b) planes perpendicular to one another, (c) lines parallel to a plane, (d) lines perpendicular to a plane, (e) pairs of skew lines.

or, Prove that if a straight line is perpendicular to each of two

intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

7. The volume of a right prism is 80 cu. ft. and its base is a triangle whose sides are 3 ft., 4 ft. and 5 ft. respectively. Find the height and the area of the total surface of the prism.

or, A conical tent is required to accommodate 4 people ; each person must have 20 sq. ft. of space on the ground and 100 cu. ft. of air to breathe. Find the height and radius of the tent. [$\pi = 22/7$]

1965

1. Show that the points $(2a, 4a)$, $(2a, 6a)$ and $(2a+\sqrt{3}a,\ 5a)$ are the vertices of an equilateral triangle whose sides are each of length $2a$. Calculate its area.

2. What do the following equations represent ?—

(*i*) $x^2+y^2-6x+8y=0$,

(*ii*) $(y+4)^2=4(x-3)$,

(*iii*) $(x-3)(y+4)=0$.

Roughly sketch the graphs.

3. Express, in the form $\frac{x}{a}+\frac{y}{b}=1$, the equation of the straight line passing through the point (3, 2) and the intersection of the lines

$$3x+y-5=0 \text{ and } x+5y+3=0.$$

Find the area of the triangle cut off from the coordinate axes by this line.

4. Prove that the centres of the three circles

$$x^2+y^2=1,\ x^2+y^2+6x-2y=6 \text{ and } x^2+y^2-12x+4y=9$$

are collinear and that their radii are in arithmetical progression.

5. Find the centre, the eccentricity, the foci and the lengths of the axes of the ellipse $9x^2+25y^2=225$.

6. (*a*) How would you measure the angle which a straight line makes with a given plane ?

(*b*) When is a straight line said to be perpendicular to a given plane ?

(*c*) When are two planes perpendicular to one another ?

(*d*) What are skew lines ?

Illustrate your answer by suitable diagrams.

or, Prove that all straight lines drawn perpendicular to a given straight line at a given point of it are coplanar.

7. The base of a right prism is a triangle whose sides are 1′9″, 1′8″ and 1′1″ long. If the height is 8′, find the whole surface and the volume of the prism.

or, Find, to the nearest tenth of a metre, the height of a conical tent which stands on a circular base of diameter 8·0 metres and which contains 90·478 cubic metres of air. [$\pi = 3{\cdot}1416$]